LOOSE PATHS IN RANDOM ORDERED HYPERGRAPHS

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ABSTRACT. We consider the length of *ordered loose paths* in the random r-uniform hypergraph $H = H^{(r)}(n,p)$. A ordered loose path is a sequence of edges $E_1, E_2, \dots, E_\ell$ where $\max\{j \in E_i\} = \min\{j \in E_{i+1}\}$ for $1 \le i < \ell$. We establish fairly tight bounds on the length of the longest ordered loose path in H that hold with high probability.

1. INTRODUCTION

There has been considerable work on the maximum length of paths in random graphs and hypergraphs, particularly in the graph case; see the survey by Frieze [2] for a summary what is known on the subject. Albert and Frieze [1] considered the maximum length of a path in an orientation of $G_{n,p}$ where the edge $\{i,j\}, i < j$ was always oriented from i to j. In this paper we consider a generalization of this problem to r-uniform hypergraphs.

Let $H = H^{(r)}(n,p)$ be the random r-uniform hypergraph on the set of vertices $[n]$ such that each r-tuple in $\binom{[n]}{r}$ is included as an edge with probability p. Let E_p denote its set of edges. Define an *ordered loose path* of *length* ℓ in H as an increasing subsequence of vertices $v_1, v_2, \dots, v_{\ell(r-1)+1} \in [n]$ such that $\{v_1 < \dots < v_r\}, \{v_r < \dots < v_{2r-1}\}, \dots, \{v_{(\ell-1)(r-1)+1} < \dots < v_{\ell(r-1)+1}\}$ are the edges of H (so that every pair of consecutive edges intersects in a single vertex). Let $\ell_{\max}$ be the maximal length of an ordered loose path in H. In the following, we discuss the likely value of $\ell_{\max}$ for varying values of p.

Theorem 1.1. *Let $r \ge 2$ and $\Omega(1) = p \le 1 - o(1)$. Then, a.a.s.*

$$\frac{(1+o(1))n}{r - p - r(1-p)^r + (pr+1)(1-p)^r p^{-2}} \le \ell_{\max} \le (1+o(1))n\left(\frac{1}{r} + \frac{1}{r(r-2+p^{-1})}\right).$$

Corollary 1.2. *Let $0 < p < 1$ be a constant. Then, for every $\varepsilon > 0$ there is an $r_0 = r_0(p,\varepsilon) \in \mathbb{N}$ such that for each $r \ge r_0$, we have a.a.s.*

$$\left(\frac{1}{r} - \varepsilon\right) n \le \ell_{\max} \le \left(\frac{1}{r} + \varepsilon\right) n.$$

Corollary 1.3. *Let $r \ge 2$ and $p = 1 - o(1)$. Then, we have a.a.s.*

$$\ell_{\max} = \frac{(1+o(1))n}{r-1}.$$

In fact, the upper bound in these corollaries is trivial (and no upper bound from Theorem 1.1 is needed), since always the length of an ordered loose path is at most

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$\frac{n}{r-1} = n\left(\frac{1}{r} + \frac{1}{r(r-1)}\right)$. However, notice that the second term of the upper bound in Theorem 1.1 is always smaller than the trivial one, since $\frac{1}{r(r-2+p^{-1})} < \frac{1}{r(r-1)}$.

Theorem 1.4. *Let $r \geq 2$ and $\sqrt{\log n}/n^{(r-1)/4} \ll p = o(1)$ or $\Omega((\log n)^{r-1}/n^{r-1}) = p \ll 1/n^{(r-1)/2}$. Then, a.a.s.*

$$\ell_{\max} = \Theta(np^{1/(r-1)}).$$

(We write $A_n \ll B_n$ (resp. $B_n \gg A_n$) if $A_n/B_n \to 0$ as $n \to \infty$.)

The next theorem fills the gap for the missing range of p from Theorem 1.4. Unfortunately, this statement (likely the lower bound) is not optimal.

Theorem 1.5. *Let $r \geq 2$ and $\Omega(1/n^{(r-1)/2}) = p = O(\sqrt{\log n}/n^{(r-1)/4})$. Then, a.a.s.*

$$\Omega(np^{1/(r-1)}/(\log n)^{1/(r-1)}) = \ell_{\max} = O(np^{1/(r-1)}).$$

The remaining (optimal) theorems describe results for the sparse regime.

Theorem 1.6. *Let $\omega = \omega(n)$ be such that $1 \ll \omega \ll \log n$ and*

$$\frac{1}{n^{r-1+1/\omega}} \leq p = O((\log n)^{r-1}/n^{r-1}).$$

Then, there exists $1 \ll \ell_0 = O(\log n)$ such that

$$\left(\frac{n}{\ell_0}\right)^{r-1+1/\ell_0} p = 1$$

and then a.a.s. $\ell_{\max} = \Theta(\ell_0)$.

Theorem 1.7. *Let $\ell \geq 2$ be an integer. Then, if $1/n^{r-1+1/\ell} \ll p \ll 1/n^{r-1+1/(\ell+1)}$, then a.a.s. $\ell_{\max} = \ell$.*

Furthermore, let $c > 0$ be a real number and $p = c/n^{r-1+1/\ell}$. Let X count the number of ordered paths of length ℓ. Then, X converges in distribution to $Po(\lambda)$, where $\lambda = c/(\ell(r-1)+1)!$.

Observe that if in Theorem 1.6 we allow $\omega = \Theta(1)$, then the lower bound on p is considered in Theorem 1.7. Also if $\omega = \Omega(\log n)$, then

$$p \geq \frac{1}{n^{r-1+1/\Omega(\log n)}} = \frac{1}{n^{r-1}} \frac{1}{n^{1/\Omega(\log n)}} = \Omega\left(\frac{1}{n^{r-1}}\right),$$

which is contained in the current range of p in Theorem 1.6.

2. Preliminaries

To bound the length of the longest ordered loose path from below, it is natural to consider a greedy algorithm, which extends an ordered loose path from its last vertex i by a hyperedge whose least vertex is i and whose "length" $j - i$, where j is its last vertex, is as small as possible.

We will use the following lemma (about random hypergraphs of edges with common left endpoints) when considering the properties of such a greedy extension algorithm.

Lemma 2.1. *Let H be a random r-hypergraph on the vertex set $\mathbb{Z}^+$ such that each edge $\{1 < i_2 < \cdots < i_{r-1} < i_r\}$ appears with probability p. Let X_i (for $i \geq 0$) be a random variable, which equals $r - 1 + i$ if (I) $\{1 < i_2 < \cdots < i_{r-1} < i_r = r + i\}$ is present in H and (II) no edge of the form $\{1 < j_2 < \cdots < j_{r-1} < j_r \leq r + i - 1\}$ is present in H, and is 0 otherwise. Define $X = \sum_{i=0}^{\infty} X_i$. Then:*

(i) The expected value equals

$$\mathbf{E}X = \sum_{i=0}^{\infty} (r - 1 + i)(1 - p)^{\binom{r-2+i}{r-1}} \left(1 - (1 - p)^{\binom{r-2+i}{r-2}}\right).$$

(ii) The expected value satisfies

$$r - p - r(1 - p)^r \leq \mathbf{E}X \leq r - p - r(1 - p)^r + (pr + 1)(1 - p)^r p^{-2}$$

and, in particular, for $p = \Omega(1)$, which is independent from r,

$$\mathbf{E}X = r - p + o_r(1),$$

where $o_r(1)$ is approaching zero as r tends to infinity.

(iii) For $p = o(1)$,

$$\mathbf{E}X = \Theta(1/p^{1/(r-1)}).$$

Proof. We divide the proof into three parts.

Part (i)**:** For $i = 0$, clearly, $\mathbf{E}X_0 = (r - 1)q_0 p_0$, where $q_0 = 1$ and $p_0 = p$.

Now consider $i = 1$. Here the edge $\{1, \ldots, r\}$ is not present (this happens with probability $q_1 = 1 - p_0 = 1 - p$) and an edge $e = \{1 < i_2 < \cdots < i_{r-1} < r + 1\}$ is present. Since we have exactly $\binom{r-1}{r-2}$ choices for vertices $i_2 < \cdots < i_{r-1}$, the latter occurs with probability $p_1 = 1 - (1 - p)^{\binom{r-1}{r-2}}$. Thus, $\mathbf{E}X_1 = rq_1 p_1$.

Assume in general that q_i is the probability of the event that no edge $\{1 < i_2 < \cdots < i_{r-1} < r + j\}$ for $0 \leq j \leq i - 1$ is present. Moreover, let p_i be the probability of the event that there is an edge of the form $\{1 < i_2 < \cdots < i_{r-1} < r + i\}$. Observe that

$$q_i = q_{i-1}(1 - p_{i-1}) \quad \text{and} \quad p_i = 1 - (1 - p)^{\binom{r-2+i}{r-2}}.$$

Thus,

$$q_i = \prod_{j=0}^{i-1} (1 - p_j) = \prod_{j=0}^{i-1} (1 - p)^{\binom{r-2+j}{r-2}} = (1 - p)^{\sum_{j=0}^{i-1} \binom{r-2+j}{r-2}} = (1 - p)^{\binom{r-2+i}{r-1}}$$

and

$$\mathbf{E}X_i = (r - 1 + i)(1 - p)^{\binom{r-2+i}{r-1}} \left(1 - (1 - p)^{\binom{r-2+i}{r-2}}\right).$$

Consequently,

$$\mathbf{E}X = \sum_{i=0}^{\infty} \mathbf{E}X_i = \sum_{i=0}^{\infty} (r - 1 + i)(1 - p)^{\binom{r-2+i}{r-1}} \left(1 - (1 - p)^{\binom{r-2+i}{r-2}}\right).$$

Part (ii)**:** We split this sum into two terms: $\mathbf{E}X_0 + \mathbf{E}X_1$ and $\sum_{i=2}^{\infty} \mathbf{E}X_i$. Observe that

$$\mathbf{E}X_0 + \mathbf{E}X_1 = (r - 1)p + r(1 - p)\left(1 - (1 - p)^{r-1}\right) = r - p - r(1 - p)^r$$

and

$$\sum_{i=2}^{\infty} \mathbf{E}X_i \le \sum_{i=2}^{\infty} (r-1+i)(1-p)^{r-2+i} = (pr+1)(1-p)^r p^{-2}.$$

Since $\mathbf{E}X_0 + \mathbf{E}X_1 \le \mathbf{E}X = \mathbf{E}X_0 + \mathbf{E}X_1 + \sum_{i=2}^{\infty} \mathbf{E}X_i$, we obtain

$$r - p - r(1-p)^r \le \mathbf{E}X \le r - p - r(1-p)^r + (pr+1)(1-p)^r p^{-2}.$$

Finally notice that both $\lim_{r\to\infty} r(1-p)^r = 0$ and $\lim_{r\to\infty}(pr+1)(1-p)^r p^{-2} = 0$ for any $p = \Omega(1)$ being independent from r. This yields $\mathbf{E}X = r - p + o_r(1)$.

Part (iii)**:** For $r = 2$ observe that

$$\mathbf{E}X = \sum_{i=0}^{\infty} (1+i)(1-p)^i p = p \sum_{i=0}^{\infty} (1+i)(1-p)^i = p \cdot \frac{1}{p^2} = 1/p,$$

as required. Therefore, we may assume that $r \ge 3$. We split the sum into two terms:

$$S_1 = \sum_{0 \le i \le (r-1)/p^{1/(r-1)}} (r-1+i)(1-p)^{\binom{r-2+i}{r-1}} \left(1 - (1-p)^{\binom{r-2+i}{r-2}}\right),$$

and

$$S_2 = \sum_{(r-1)/p^{1/(r-1)} < i} (r-1+i)(1-p)^{\binom{r-2+i}{r-1}} \left(1 - (1-p)^{\binom{r-2+i}{r-2}}\right).$$

Observe that

$$\begin{aligned}
S_1 &\le \sum_{0 \le i \le (r-1)/p^{1/(r-1)}} (r-1+i) \left(1 - (1-p)^{\binom{r-2+i}{r-2}}\right) \\
&\le \sum_{0 \le i \le (r-1)/p^{1/(r-1)}} (r-1+i) \left(1 - \left(1 - p\binom{r-2+i}{r-2}\right)\right) \\
&= p \sum_{0 \le i \le (r-1)/p^{1/(r-1)}} (r-1+i) \binom{r-2+i}{r-2} \\
&\le p \sum_{0 \le i \le (r-1)/p^{1/(r-1)}} (r-1+i)(r-2+i)^{r-2} \\
&\le p \cdot (r-1)/p^{1/(r-1)} (r-1+(r-1)/p^{1/(r-1)})^{r-1} = O(1/p^{1/(r-1)}).
\end{aligned}$$

Now

$$\begin{aligned}
S_2 &= \sum_{(r-1)/p^{1/(r-1)} < i} (r-1+i)(1-p)^{\binom{r-2+i}{r-1}} \left(1 - (1-p)^{\binom{r-2+i}{r-2}}\right) \\
&\le \sum_{(r-1)/p^{1/(r-1)} < i} (r-1+i) \exp\left\{-p\binom{r-2+i}{r-1}\right\} \left(1 - \left(1 - p\binom{r-2+i}{r-2}\right)\right) \\
&= p \sum_{(r-1)/p^{1/(r-1)} < i} (r-1+i) \binom{r-2+i}{r-2} \exp\left\{-p\binom{r-2+i}{r-1}\right\} \\
&\le p \sum_{(r-1)/p^{1/(r-1)} < i} (r-1+i) \left(\frac{e(r-2+i)}{r-2}\right)^{r-2} \exp\left\{-p\left(\frac{r-2+i}{r-1}\right)^{r-1}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= p\left(\frac{e}{r-2}\right)^{r-2} \sum_{(r-1)/p^{1/(r-1)}<i} (r-1+i)(r-2+i)^{r-2} \exp\left\{-p\left(\frac{1}{r-1}\right)^{r-1}(r-2+i)^{r-1}\right\}\\
&= p\left(\frac{e}{r-2}\right)^{r-2} \sum_{(r-1)/p^{1/(r-1)}+r-2<i} (i+1)i^{r-2} \exp\left\{-p\left(\frac{1}{r-1}\right)^{r-1} i^{r-1}\right\}\\
&\le p\left(\frac{e}{r-2}\right)^{r-2} \sum_{(r-1)/p^{1/(r-1)}+r-2<i} (2i)i^{r-2} \exp\left\{-p\left(\frac{1}{r-1}\right)^{r-1} i^{r-1}\right\}\\
&= 2p\left(\frac{e}{r-2}\right)^{r-2} \sum_{(r-1)/p^{1/(r-1)}+r-2<i} i^{r-1} \exp\left\{-p\left(\frac{1}{r-1}\right)^{r-1} i^{r-1}\right\}.
\end{aligned}
$$

Consider $f(x) = x^{r-1}\exp(-cx^{r-1})$ for $x \ge 0$. It is easy to check that f is positive and takes the maximum at point $1/c^{1/(r-1)}$ and it is decreasing function for $x \ge 1/c^{1/(r-1)}$. Furthermore,

$$
\int_0^\infty x^{r-1}\exp(-cx^{r-1})dx = \frac{\Gamma(r/(r-1))}{c^{r/(r-1)}(r-1)},
$$

where $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}$ is the *gamma function* (see integral 3.326 in [4]). Let $c = p\left(\frac{1}{r-1}\right)^{r-1}$. Then,

$$
1/c^{1/(r-1)} = (r-1)/p^{1/(r-1)}.
$$

Thus,

$$
\begin{aligned}
S_2 &\le 2p\left(\frac{e}{r-2}\right)^{r-2} \sum_{(r-1)/p^{1/(r-1)}+r-2<i} i^{r-1} \exp\left\{-p\left(\frac{1}{r-1}\right)^{r-1} i^{r-1}\right\}\\
&\le 2p\left(\frac{e}{r-2}\right)^{r-2} \int_0^\infty x^{r-1} \exp\left\{-p\left(\frac{1}{r-1}\right)^{r-1} x^{r-1}\right\}\\
&= 2p\left(\frac{e}{r-2}\right)^{r-2} \frac{\Gamma(r/(r-1))}{c^{r/(r-1)}(r-1)}\\
&= 2p\left(\frac{e}{r-2}\right)^{r-2} \frac{(r-1)^{r-1}\Gamma(r/(r-1))}{p^{r/(r-1)}} = O(1/p^{1/(r-1)}).
\end{aligned}
$$

Hence,

$$
\mathbf{E}X = S_1 + S_2 = O(1/p^{1/(r-1)}).
$$

It remains to show the lower bound. Recall that

$$
(1+x)^k \le 1 + \frac{kx}{1-(k-1)x} = \frac{1+x}{1-(k-1)x} \quad \text{for} \quad -1 \le x < \frac{1}{k-1} \quad \text{and} \quad k \ge 1.
$$

Thus,

$$
1-(1-p)^{\binom{r-2+i}{r-2}} \ge \left(1 - \left(1 - \frac{p\binom{r-2+i}{r-2}}{1+p\left(\binom{r-2+i}{r-2}-1\right)}\right)\right) = \frac{p\binom{r-2+i}{r-2}}{1+p\left(\binom{r-2+i}{r-2}-1\right)}.
$$

r	$\mathbf{E}X$
2	2
3	2.6416...
4	3.5634...
5	4.5312...
6	5.5156...
7	6.5078...
8	7.5039...
9	8.5019...
10	9.5009...

TABLE 1. $\mathbf{E}X$ from Lemma 2.1 for $p = 1/2$ and $r \in [10]$.

Moreover, for $p = o(1)$ and $0 \le i \le 1/p^{1/(r-1)}$ we get

$$1 + p\left(\binom{r-2+i}{r-2} - 1\right) \le 1 + p(r-2+i)^{r-2} = O(p^{1/(r-1)}) = o(1) \le 2.$$

Hence,

$$1 - (1-p)^{\binom{r-2+i}{r-2}} \ge p\binom{r-2+i}{r-2}/2.$$

Also, since $1 - x \ge e^{-2x}$ for $0 \le x \le 1/2$, we obtain for $p = o(1)$ and $0 \le i \le 1/p^{1/(r-1)}$

$$(1-p)^{\binom{r-2+i}{r-1}} \ge \exp\left\{-2p\binom{r-2+i}{r-1}\right\} \ge \exp\left\{-2p(r-2+i)^{r-1}\right\} = \Omega(1).$$

Therefore,

$$\begin{aligned}\mathbf{E}X &\ge \Omega(p) \sum_{0 \le i \le 1/p^{1/(r-1)}} (r-1+i)\binom{r-2+i}{r-2} \\ &\ge \Omega(p) \sum_{0 \le i \le 1/p^{1/(r-1)}} i^{r-1} \\ &\ge \Omega(p)\frac{1}{p^{1/(r-1)}}\left(\frac{1}{p^{1/(r-1)}}\right)^{r-1} = \Omega(1/p^{1/(r-1)})\end{aligned}$$

and, finally yielding $\mathbf{E}X = \Theta(1/p^{1/(r-1)})$. □

In Table 1 we present the specific values of this expectation in case when $p = 1/2$ and $r \in [10]$.

Remark 1. For $r = 2$ the random variable i has the geometric distribution with probability of success p. Indeed, in this case $X_i = 1 + i$ can be viewed as i failures (occurring with probability $q_i = (1-p)^i$) before the first success happens (with probability $p_i = p$) for each $i \ge 0$.

We will also need an easy auxiliary result.

Claim 2.2. *Let $p = \Omega((\log n)/n^{r-1})$ and*

$$a = \begin{cases} \frac{4}{p}\log n & \text{if } r = 2, \\ \left(\frac{4(r-2)^{r-2}}{p}\log n\right)^{1/(r-1)} & \text{if } r \geq 3. \end{cases}$$

Then, for $r = 2$ and any disjoint subsets $A_1, A_2 \subseteq [n]$ with $|A_1| = |A_2| = a \geq 10$, with probability $1 - O(n^{-10})$ there is in $H^{(2)}(n,p)$ an edge e such that $|A_1 \cap e| = |A_2 \cap e| = 1$. Moreover, for $r \geq 3$ and any disjoint subsets $A_1, A_2, A_3 \subseteq [n]$ with $|A_1| = |A_2| = |A_3| = a$, with probability $1 - O(n^{-10})$ there is in $H^{(r)}(n,p)$ an edge e such that $|A_1 \cap e| = |A_2 \cap e| = 1$ and $|A_3 \cap e| = r - 2$.

Proof. We prove the statement for $r \geq 3$. (The case $r = 2$ is very similar.) Observe that the probability that there are A_1, A_2, A_3 with no edge described above is at most

$$\begin{aligned}
\binom{n}{a}^3 (1-p)^{a^2\binom{a}{r-2}} &\leq n^{3a} \exp\left\{-pa^2\binom{a}{r-2}\right\} \\
&= \exp\left\{3a\log n - pa^2\binom{a}{r-2}\right\} \\
&\leq \exp\left\{3a\log n - pa^2a^{r-2}/(r-2)^{r-2}\right\} \\
&= \exp\left\{a\left(3\log n - pa^{r-1}/(r-2)^{r-2}\right)\right\} \\
&\leq \exp\left\{a(3\log n - 4\log n)\right\} = o(1).
\end{aligned}$$

□

We will need the following concentration result of Warnke [5]:

Lemma 2.3 (Warnke [5])**.** *Let $W = (W_1, W_2, \ldots, W_n)$ be a family of independent random variables with W_i taking values in a set Λ_i. Let $\Omega = \prod_{i\in[n]} \Lambda_i$ and suppose that $\Gamma \subseteq \Omega$ and $f : \Omega \to \mathbf{R}$ are given. Suppose also that whenever $\mathbf{x}, \mathbf{x}' \in \Omega$ differ only in the i-th coordinate*

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq \begin{cases} c_i & \text{if } \mathbf{x} \in \Gamma. \\ d_i & \text{otherwise.} \end{cases}$$

If $Y = f(X)$, then for all reals $\gamma_i > 0$,

$$\mathbf{Pr}(Y \geq \mathbf{E}(Y) + t) \leq \exp\left\{-\frac{t^2}{2\sum_{i\in[n]}(c_i + \gamma_i(d_i - c_i))^2}\right\} + \mathbf{Pr}(W \notin \Gamma)\sum_{i\in[n]}\gamma_i^{-1}. \qquad (1)$$

3. Lower bound for $\sqrt{\log n}/n^{(r-1)/4} \ll p \leq 1 - o(1)$.

We present a lower bound by applying a greedy algorithm. This will imply the lower bound in Theorem 1.1 and Theorem 1.4 for $\sqrt{\log n}/n^{(r-1)/4} \ll p = o(1)$.

Let $e = \{i_1 < \cdots < i_r\}$ be an edge. We define the *length* of e as $\ell(e) = i_r - i_1$. We will greedily construct a loose path $e_1, e_2, \ldots$. We start at the vertex 1 and choose the first present edge $e_1 = \{1 < i_2 < \cdots < i_{r-1} < i_r\}$, that means, an edge of the smallest length that contains vertex 1. Now we repeat the process starting at the vertex i_r and find the first edge e_2 that starts at i_r, etc. Let L_i be a random variable that counts the length of edge e_i. Thus, we want to analyze the random variable K such that

$$L_1 + L_2 + \cdots + L_K \le n-1 \quad \text{and} \quad L_1 + L_2 + \cdots + L_{K+1} > n-1.$$

Since $L_i = X$, where X is a random variable defined in Lemma 2.1 (below), we get

$$(\mathbf{E}K)(\mathbf{E}X) \le n-1 \quad \text{and} \quad (\mathbf{E}(K+1))(\mathbf{E}X) > n-1$$

implying that $\mathbf{E}K \sim n/(\mathbf{E}X)$.

Let the random variable Y denote the maximum length of an ordered path in $H^{(r)}(n,p)$. Clearly, $Y \ge K$ and $\mathbf{E}Y \ge \mathbf{E}K$. We will show by using Lemma 2.3 that Y is tightly concentrated around its expectation. This will imply that a.a.s. $Y \ge (1+o(1))\mathbf{E}K$. We will have to estimate $|Y(W_1,\dots,W_i,\dots,W_n) - Y(W_1,\dots,W_i',\dots,W_n)|$ for all W, i, W_i'.

Now we will check the assumptions of Lemma 2.3. Let W_i be the set of edges of $H^{(r)}(n,p)$ that contain i and are contained in $[i-1]$.

For a given value of $Y = Y(W_1,\dots,\dots,W_n) = y$ let P be a loose path of length y. Suppose first that $i \in P$ and that edge $e \in (P \cap W_i) \setminus W_i'$. By removing edge e we split P into two paths P_1 and P_2 such that $V(P_1) \subseteq [i-r+1]$ and $V(P_2) \subseteq \{i+r-1,\dots,n\}$. Define $c = 3a$, where a is defined in Claim 2.2. If $|E(P_1)| \le c$ or $|E(P_2)| \le c$, then, clearly, $|Y(W_1,\dots,W_i,\dots,W_n) - Y(W_1,\dots,W_i',\dots,W_n)| \le c$.

Suppose that $|E(P_1)| \ge c$ and $|E(P_2)| \ge c$. Let $m_i = |E(P_i)|$, where clearly $m_1 + m_2 = y-1$. Let $E(P_i) = \{e_1^i,\dots,e_{m_i}^i\}$, where e_j^i precedes e_{j+1}^i. Denote by $first(e)$ and $last(e)$ the first and the last vertex of edge e, respectively. Define

$$A_1 = \{last(e_j^1) : m_1 - 2a + 1 \le j \le m_1 - a\} \quad \text{and} \quad A_2 = \{first(e_j^2) : 1 \le j \le a\}$$

and A_3 be any subset of $\bigcup_{j=m_1-a+1}^{m_1} e_j^1$ of size a. Due to Claim 2.2 there exists an edge f such that $f = \{last(e_{j_1}^1) < \cdots < first(e_{j_2}^2)\}$ with $m_1 - 2a + 1 \le j_1 \le m_1 - a$ and $1 \le j_2 \le a$. Thus, P_1 and P_2 together with f contain a path of length at least $y - c$ yielding $|Y(W_1,\dots,W_i,\dots,W_n) - Y(W_1,\dots,W_i',\dots,W_n)| \le c$.

It remains to consider the case in which i is not in P and assume that W_i' consists of all possible edges. By adding these edges we can create a longer path. Clearly, here we can argue by the symmetric argument that we cannot create a path longer that $k + c$.

Now we apply Lemma 2.3 with Γ equal to the set of W satisfying the conditions in Claim 2.2 and $c_i = c$ for $W \in \Gamma$ and $d_i = n$ and $\gamma_i = n^{-3}$, where

$$c_i = c = 3a \begin{cases} \frac{12}{p} \log n & \text{if } r = 2, \\ \left(\frac{12(r-2)^{r-2}}{p} \log n \right)^{1/(r-1)} & \text{if } r \ge 3. \end{cases}$$

Thus, for $p \ge \omega\sqrt{\log n}/n^{(r-1)/4}$ and $t = (\mathbf{E}Y)/\omega^{1/(r-1)}$ we get

$$\begin{aligned} \mathbf{Pr}\left(|Y - \mathbf{E}Y| \ge \frac{\mathbf{E}Y}{\omega^{1/(r-1)}} \right) &\le 2\exp\left\{ -\frac{(\mathbf{E}Y)^2}{2\omega^{2/(r-1)}(nc^2 + n^{-1})} \right\} + n^{-7} \\ &\le 2\exp\left\{ -\frac{(\mathbf{E}K)^2}{2\omega^{2/(r-1)}(nc^2 + n^{-1})} \right\} + n^{-7}. \end{aligned}$$

Now notice that the order of the magnitude of the exponent satisfies

$$\frac{(\mathbf{E}K)^2}{\omega^{2/(r-1)}nc^2} \geq \frac{n}{2\omega^{2/(r-1)}c^2(\mathbf{E}X)^2} = \frac{n}{2\omega^{2/(r-1)}\left(\frac{\log n}{p}\right)^{2/(r-1)}\left(\frac{1}{p}\right)^{2/(r-1)}} = \frac{np^{4/(r-1)}}{2\omega^{2/(r-1)}(\log n)^{2/(r-1)}}.$$

Hence, for $p \geq \omega\sqrt{\log n}/n^{(r-1)/4}$,

$$\frac{(\mathbf{E}Y)^2}{\omega^{2/(r-1)}nc^2} \geq \frac{\omega^{2/(r-1)}}{2}$$

implying that $\mathbf{Pr}(|Y - \mathbf{E}Y| \geq (\mathbf{E}Y)/\omega^{1/(r-1)}) = o(1)$ and so a.a.s. $Y \sim \mathbf{E}Y$.

4. Lower bound for $\Omega(1/n^{(r-1)/2}) = p = O(\sqrt{\log n}/n^{(r-1)/4})$.

Let $\ell = O(n(p/\log n)^{1/(r-1)})$. We will greedily build a path of length ℓ by applying Claim 4.1 (below). Observe that

$$d := \frac{n}{\ell} = \Omega\left(\frac{n}{n(p/\log n)^{1/(r-1)}}\right) = \Omega\left(\left(\frac{\log n}{p}\right)^{1/(r-1)}\right).$$

First we reveal all r-tuples starting at vertex 1. By Claim 4.1 there is an edge e_1 of length at most d starting at this vertex. Now we reveal all r-tuples that start at the last vertex of e_1, etc. This way we see that a.a.s. there is a path of length at least ℓ.

Claim 4.1. *Let $p = \Omega((\log n)/n^{r-1})$ and $(r-1)\left(\frac{2\log n}{p}\right)^{1/(r-1)} \leq d \leq n$. Then, a.a.s. for each $i \in [n-d]$ there is an edge $e \in E_p$ of length at most d that starts at i. That means there is a vertex $j \leq i + d$ such $e = \{i < i_2 < \cdots < i_{r-1} < j\} \in E_p$.*

Proof. Observe that for a fixed vertex i, the probability that there is no edge of the form $\{i < i_2 < \cdots < i_{r-1} < i_r\}$ with $i_r \leq i + d$ is exactly $(1-p)^{\binom{d}{r-1}}$. Hence, the probability that there exists a vertex i with no such edges is at most

$$n(1-p)^{\binom{d}{r-1}} \leq \exp\left\{\log n - p\binom{d}{r-1}\right\} \leq \exp\left\{\log n - p\left(\frac{d}{r-1}\right)^{r-1}\right\} = o(1).$$

□

5. Lower bound for $\Omega((\log n)^{r-1}/n^{r-1}) = p \ll 1/n^{(r-1)/2}$.

Let $\ell = \frac{1}{8(r-1)}np^{1/(r-1)} - \frac{1}{r-1}$. We will apply the second moment method. For a fixed $S \in \binom{[n]}{\ell(r-1)+1}$, let X_S be an indicator random variable which equals 1 if S induces an ordered path of length ℓ denoted by P_S; otherwise 0. Define $X = \sum_{S \in \binom{[n]}{\ell(r-1)+1}} X_S$. Clearly,

$$\begin{aligned}\mathbf{E}X &= \binom{n}{\ell(r-1)+1}p^\ell \geq \left(\frac{n}{\ell(r-1)+1}\right)^{\ell(r-1)+1}p^\ell \\ &\geq \left(\frac{n}{\ell(r-1)+1}\right)^{\ell(r-1)}p^\ell = \left(\left(\frac{n}{\ell(r-1)+1}\right)^{r-1}p\right)^\ell = 8^{\ell(r-1)},\end{aligned}$$

which tends to infinity.

Now we calculate $\sum_{S\neq S'} \mathbf{E}(X_S X_S')$ for $s := \ell(r-1)+1 = |S| = |S'|$. First observe that

$$\sum_{|S\cap S'|=0} \mathbf{E}(X_S X_{S'}) \le \binom{n}{s} p^\ell \binom{n}{s} p^\ell = (\mathbf{E}X)^2.$$

Similarly,

$$\sum_{1\le |S\cap S'|\le r-1} \mathbf{E}(X_S X_S') \le \binom{n}{s} p^\ell \binom{n}{s-1} p^\ell = \binom{n}{s} p^\ell \binom{n}{s} \frac{s}{n-s+1} p^\ell$$
$$= (\mathbf{E}X)^2 \frac{s}{n-s+1} = o((\mathbf{E}X)^2).$$

It remains to consider the case when the paths induced by S and S' share some edges. Let $n_{a,b}$ be the number of ordered pairs (S,S') such that $|E(P_S)\cap E(P_{S'})| = a$ and the number of vertices of degree 2 induced by $E(P_S)\cap E(P_{S'})$ is exactly b. For example, if $E(P_S)\cap E(P_{S'})$ is just one path, then $b = a-1$, or if $E(P_S)\cap E(P_{S'})$ is a matching of size a, then $b = 0$. In general, it is easy to observe that $E(P_S)\cap E(P_{S'})$ consists of $c := a-b$ vertex disjoint paths.

First we will consider the following problem. Let P be a path of length ℓ. We calculate in how many ways we can choose vertex-disjoint paths $P_1,\dots,P_c$ contained in P such that $|E(P_1)| + \dots + |E(P_c)| = a$ (this will also imply that the number of vertices of degree 2 in P_i's is exactly b). Let $x_i = |E(P_i)|$ and y_i be a gap (in terms of the number of edges) between P_i and P_{i+1}. Any choice of P_i's can be encoded as an integer solution of

$$y_0 + x_1 + y_1 + x_2 + y_2 + \dots + x_c + y_c = \ell,$$

where $x_i \ge 1$ for $1\le i\le c$ and $y_0\ge 0$, $y_c \ge 0$, $y_j\ge 1$ for $1\le j\le c-1$. This is equivalent of solving

$$x_1 + \dots + x_c = a \quad \text{and} \quad y_0 + \dots + y_c = \ell - a,$$

which turns out to be equivalent to solving

$$(x_1-1)+\dots+(x_c-1) = a-c \;\text{ and }\; y_0+(y_1-1)+(y_2-1)+\dots+(y_{c-1}-1)+y_c = \ell-a-c+1$$

with all (x_i-1) (for $1\le i\le c$), y_0, y_c and (y_j-1) (for $1\le j\le c-1$) nonnegative. Thus, the number of integer solutions is exactly

$$\binom{a-1}{c-1}\binom{\ell-a+1}{c} \le 2^a \ell^c = 2^a \ell^{a-b}.$$

Now notice that where $s = \ell(r-1)+1$,

$$\begin{aligned} n_{a,b} &\le \binom{n}{s}\cdot 2^a\ell^{a-b}\binom{n}{s-(ar-b)} \\ &= \binom{n}{s}\cdot 2^a\ell^{a-b}\binom{n}{s}\frac{s-(ar-b)+1}{n-s+1}\cdots\frac{s-1}{n-s+(ar-b-1)}\frac{s}{n-s+(ar-b)} \\ &\le \binom{n}{s}\binom{n}{s}2^a\ell^{a-b}\left(\frac{s}{n-s}\right)^{ar-b}. \end{aligned}$$

Hence,

$$\sum_{1\le a\le \ell}\sum_{0\le b\le a-1} n_{a,b}p^{2\ell-a} \le \sum_{1\le a\le \ell}\sum_{0\le b\le a-1} (\mathbf{E}X)^2 2^a \ell^{a-b}\left(\frac{s}{n-s}\right)^{ar-b} p^{-a}$$
$$= (\mathbf{E}X)^2 \sum_{1\le a\le \ell} 2^a\ell^a\left(\frac{s}{n-s}\right)^{ar} p^{-a} \sum_{0\le b\le a-1} \ell^{-b}\left(\frac{s}{n-s}\right)^{-b}.$$

Since $p \ll 1/n^{(r-1)/2})$, $\ell^2 \ll n$ and so $\ell s \ll n-s$. Thus, $\frac{n-s}{\ell s} - 1 \ge \frac{n}{2\ell s}$ and

$$\sum_{0\le b\le a-1} \ell^{-b}\left(\frac{s}{n-s}\right)^{-b} = \sum_{0\le b\le a-1}\left(\frac{n-s}{\ell s}\right)^{b} = \frac{\left(\frac{n-s}{\ell s}\right)^a - 1}{\frac{n-s}{\ell s}-1} \le \frac{\left(\frac{n-s}{\ell s}\right)^a}{\frac{n}{2\ell s}}.$$

Hence,

$$\sum_{1\le a\le \ell}\sum_{0\le b\le a-1} n_{a,b}p^{2\ell-a} \le (\mathbf{E}X)^2\frac{2\ell s}{n}\sum_{1\le a\le \ell} 2^a\ell^a\left(\frac{s}{n-s}\right)^{ar} p^{-a}\left(\frac{n-s}{\ell s}\right)^a$$
$$= (\mathbf{E}X)^2\frac{2\ell s}{n}\sum_{1\le a\le \ell} 2^a\left(\frac{s}{n-s}\right)^{ar-a} p^{-a}$$
$$= (\mathbf{E}X)^2\frac{2\ell s}{n}\sum_{1\le a\le \ell} \left(2\left(\frac{s}{n-s}\right)^{r-1} p^{-1}\right)^a. \tag{2}$$

Recall that $s = \ell(r-1)+1 = \frac{1}{8(r-1)}np^{1/(r-1)}$ and $s \ll n$. Thus,

$$2\left(\frac{s}{n-s}\right)^{r-1}p^{-1} \le 2\left(\frac{2s}{n}\right)^{r-1}p^{-1} = 2\left(\frac{1}{4(r-1)}\right)^{r-1} \le 1/2$$

and

$$\sum_{1\le a\le \ell}\left(2\left(\frac{s}{n-s}\right)^{r-1}p^{-1}\right)^a \le \sum_{1\le a\le \ell} 1/2^a \le 1.$$

Consequently,

$$\sum_{1\le a\le \ell}\sum_{0\le b\le a-1} n_{a,b}p^{2\ell-a} \le (\mathbf{E}X)^2\frac{2\ell s}{n} = o((\mathbf{E}X)^2).$$

Summarizing, we have shown that

$$\frac{\mathbf{E}X^2}{(\mathbf{E}X)^2} = 1+o(1),$$

that means, if $\ell = \frac{1}{8(r-1)}np^{1/(r-1)} - \frac{1}{r-1}$, then a.a.s. there is a path of length ℓ.

6. Upper bound for $\Omega((\log n)^{r-1}/n^{r-1}) = p$.

Here we present two general upper bounds. The first one is better in the case when p is bounded from below by a constant – implying the bound in Theorem 1.1. The second approach gives the upper bounds in Theorems 1.4 and 1.5.

6.1. **Bounding the union of paths.** Let P be an ordered loose path with ℓ edges. For an integer i satisfying $r-1 \le i \le n-1$, let x_i count the number of edges of length of i in P. Clearly, $\ell = x_{r-1} + x_r + \cdots + x_{n-1}$. Observe that

$$n-1 \ge (r-1)x_{r-1} + rx_r + \cdots + (n-1)x_{n-1} \ge (r-1)x_{r-1} + r(\ell - x_{r-1}).$$

Thus,

$$\ell \le \frac{n-1+x_{r-1}}{r}.$$

Now we will bound x_{r-1} from above by using a greedy approach. Let H be an r-uniform hypergraph on the set of vertices $[n]$ (not necessary random) and P an ordered path. Let Q be a subgraph of P that consists of edges of length $r-1$ only. That means Q is a union of loose paths and $|E(Q)| = x_{r-1}$. Assume that there are edges $e \in E(H) \setminus E(Q)$ with $length(e) = r-1$ and $f \in E(Q)$ such that e proceeds before f and $E(P) \setminus e \cup f$ is still a union of loose paths. Now we replace f by e obtaining a new union of loose paths R of length x_{r-1}. We can repeat this process until such e and f no longer exist and assume that R has this property.

Let $F = \emptyset$. We start at vertex 1 and check if $\{1 < 2 < \cdots < r\}$ is present in H. If so then we accept it and add it to F. Assume that we already have chosen $F = \{f_1, \ldots, f_i\}$ that creates a union of loose paths. Next we consider the next available edge of length $r-1$ the occurs after f_i and check whether this edge together with F is still a union of loose paths. If so, then we accept it. Otherwise, we repeat the process. Observe that at the end $F = E(R)$.

Let Y be the random variable that counts the number of edges in the largest union of loose paths in $H^{(r)}(n,p)$. We will show that Y is concentrated around its mean. This together with Lemma 6.1 (below) will imply that a.a.s.

$$x_{r-1} \le Y = (1+o(1))n/((r-2)+p^{-1}).$$

Now we show by using McDiarmid's inequality[1] that $Y \sim \mathbf{E}Y$. Choose

$$t = \frac{\mathbf{E}Y}{\omega} = \frac{n}{\omega((r-2)+p^{-1})} \quad \textit{and} \quad c_i = 2,$$

since by removing one vertex (or adding), we change the size of a union of loose paths by at most two. Thus,

$$\mathbf{Pr}(|Y - \mathbf{E}Y| \ge t) \le 2\exp\left\{-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right\} = 2\exp\left\{-\frac{n}{4\omega^2((r-2)+p^{-1})^2}\right\} = o(1)$$

for $p = \Omega(1)$.

Lemma 6.1. *Let H be a random r-hypergraph on the vertex set $\mathbb{Z}^+$ such that each edge $\{j+1 < j+2 < \cdots < j+r\}$ (for $j \ge 0$) appears with probability p. Let X_i (for $i \ge 0$) be a random variable, which equals $r-1+i$ if $\{i+1 < \cdots < i+r\}$ is the first present edge of length $r-1$ that appears after vertex 1. Define $X = \sum_{i=0}^{\infty} X_i$. Then,*

$$\mathbf{E}X = \sum_{i=0}^{\infty}(r-1+i)(1-p)^i p = r-2+p^{-1}.$$

[1]This can be viewed as Warnke's inequality (from Lemma 2.3) with $\Gamma = \Omega$.

Proof. It suffices to observe that $\mathbf{E}X_i = (r-1+i)(1-p)^i p$, since none of the edges $\{j < \cdots < j+r-1\}$ for $j \in [i]$ is present in H with probability $1-p$, for each missing edge. □

6.2. **First moment method.** Let X count the number of paths of length $\ell = \frac{4^{1/(r-1)}e}{r-1}np^{1/(r-1)}$. Then,

$$\begin{aligned}\mathbf{E}X &= \binom{n}{\ell(r-1)+1}p^\ell \le \left(\frac{en}{\ell(r-1)}\right)^{\ell(r-1)+1} p^\ell \\ &= \left(\left(\frac{en}{\ell(r-1)}\right)^{(r-1)} p\left(\frac{en}{\ell(r-1)}\right)^{1/\ell}\right)^\ell \\ &\le \left(\left(\frac{en}{\ell(r-1)}\right)^{(r-1)} pn^{1/\ell}\right)^\ell = \left(\frac{1}{4}n^{1/\ell}\right)^\ell.\end{aligned}$$

If $p \ge (\frac{r-1}{4^{1/(r-1)}e})^{r-1}(\log n)^{r-1}/n^{r-1}$ and therefore $\ell = \frac{4^{1/(r-1)}e}{r-1}np^{1/(r-1)} \ge \log n$, we get

$$\mathbf{E}X \le \left(\frac{1}{4}n^{1/\ell}\right)^\ell = \left(\frac{1}{4}n^{1/\log n}\right)^\ell = \left(\frac{e}{4}\right)^\ell = o(1).$$

Thus, if $\ell \ge \frac{4^{1/(r-1)}e}{r-1}np^{1/(r-1)}$, then a.a.s. there is no path of length ℓ.

7. Lower and upper bound for $\frac{1}{n^{r-1+1/\omega}} \le p \le O((\log n)^{r-1}/n^{r-1})$.

Here we will prove Theorem 1.6. The first part of the statement is proved in the following claim.

Claim 7.1. *There exists* $1 \ll \ell_0 = O(\log n)$ *such that*

$$\left(\frac{n}{\ell_0}\right)^{r-1+1/\ell_0} p = 1.$$

Proof. For $x > 0$ define

$$f(x) = \left(\frac{n}{x}\right)^{r-1+1/x} p.$$

Clearly, f is continuous function. We will show that there are $1 \ll x_1 \le x_2 = O(\log n)$ such that $f(x_1) > 1$ and $f(x_2) < 1$. Thus, the Intermediate Value Theorem will imply the statement.

Recall that $\frac{1}{n^{r-1+1/\omega}} \le p$, where $1 \ll \omega \ll \log n$. Let $x_1 = \log \omega$. Note that

$$f(x_1) = \left(\frac{n}{x_1}\right)^{r-1+1/x_1} p \ge \left(\frac{n}{\log\omega}\right)^{r-1+1/\log\omega} \frac{1}{n^{r-1+1/\omega}} = \frac{n^{1/\log\omega - 1/\omega}}{(\log w)^{r-1+1/\omega}} \ge \frac{n^{1/\log\omega-1/\omega}}{(\log w)^r}.$$

For large n, we have

$$1/\log\omega - 1/\omega \ge 1/(2\log\omega) \ge 1/(2\log\log n).$$

Thus,

$$f(x_1) \ge \frac{n^{1/(2\log\log n)}}{(\log\log n)^r} = \exp\left\{\frac{\log n}{2\log\log n} - r\log\log\log n\right\},$$

which tends to infinity. Hence, $f(x_1) > 1$.

Now let $C > 0$ be an arbitrarily large constant and assume that $p \le \frac{C(\log n)^{r-1}}{n^{r-1}}$. Let $d > 1$ be (sufficiently large) such that

$$\frac{Ce^{1/d}}{d^{r-1}} < 1.$$

Define $x_2 = d\log n$ and observe that

$$\begin{aligned} f(x_2) &\le \left(\frac{n}{d\log n}\right)^{r-1+1/(d\log n)} \frac{C(\log n)^{r-1}}{n^{r-1}} \\ &= \frac{C}{d^{r-1+1/(d\log n)}} \left(\frac{n}{\log n}\right)^{1/(d\log n)} \le \frac{C}{d^{r-1}} n^{1/(d\log n)} = \frac{Ce^{1/d}}{d^{r-1}} < 1. \end{aligned}$$

□

7.1. **Upper bound.** Let $\ell = \frac{2e(r-1)\ell_0 - 1}{r-1}$. Hence, $1/\ell \le 1/\ell_0$ and

$$\begin{aligned} \mathbf{E}X &= \binom{n}{\ell(r-1)+1} p^\ell \le \left(\frac{en}{\ell(r-1)+1}\right)^{\ell(r-1)+1} p^\ell \\ &= \left(\left(\frac{n}{2(r-1)\ell_0}\right)^{r-1+1/\ell} p\right)^\ell \le \left(\left(\frac{n}{2(r-1)\ell_0}\right)^{r-1+1/\ell_0} p\right)^\ell \\ &= \left(\frac{1}{(2(r-1))^{r-1+1/\ell_0}} \left(\frac{n}{\ell_0}\right)^{r-1+1/\ell_0} p\right)^\ell = \frac{1}{(2(r-1))^{\ell(r-1)+\ell/\ell_0}} \cdot 1 = o(1). \end{aligned}$$

7.2. **Lower bound.** Let $\ell = \frac{\ell_0/4 - 1}{r-1}$. Hence, $1/\ell \ge 1/\ell_0$ and

$$\begin{aligned} \mathbf{E}X &= \binom{n}{\ell(r-1)+1} p^\ell \ge \left(\frac{n}{\ell(r-1)+1}\right)^{\ell(r-1)+1} p^\ell \\ &= \left(\left(\frac{4n}{\ell_0}\right)^{r-1+1/\ell} p\right)^\ell \ge \left(\left(\frac{4n}{\ell_0}\right)^{r-1+1/\ell_0} p\right)^\ell \\ &= \left(4^{r-1+1/\ell_0} \left(\frac{n}{\ell_0}\right)^{r-1+1/\ell_0} p\right)^\ell = 4^{\ell(r-1)+\ell/\ell_0} \cdot 1 \gg 1. \end{aligned}$$

We will now modify the calculations from Section 5. Since $s = \Theta(\ell) = O(\log n)$, we bound (2) as follows:

$$\begin{aligned} 2\left(\frac{s}{n-s}\right)^{r-1} p^{-1} &\le 2\left(\frac{2s}{n}\right)^{r-1} p^{-1} = 2\left(\frac{2(\ell(r-1)+1)}{n}\right)^{r-1} p^{-1} \\ &= 2\left(\frac{\ell_0}{2n}\right)^{r-1} p^{-1} = \frac{1}{2^{r-2}}\left(\frac{n}{\ell_0}\right)^{1/\ell_0} \le n^{1/\ell_0}. \end{aligned}$$

Consequently,

$$\sum_{1\le a\le \ell} \sum_{0\le b\le a-1} n_{a,b} p^{2\ell-a} \le (\mathbf{E}X)^2 \frac{2\ell s}{n} \sum_{1\le a\le \ell} \left(2\left(\frac{s}{n-s}\right)^{r-1} p^{-1}\right)^a$$

$$\leq (\mathbf{E}X)^2 \frac{2\ell s}{n} \sum_{1\leq a\leq \ell} \left(n^{1/\ell_0}\right)^a \leq (\mathbf{E}X)^2 \frac{2\ell s}{n} O(1) \left(n^{1/\ell_0}\right)^\ell .$$

Since $\ell/\ell_0 < 1/(4(r-1))$ and $\ell s = O(\log^2 n)$, we get that

$$\begin{aligned}\sum_{1\leq a\leq \ell} \sum_{0\leq b\leq a-1} n_{a,b} p^{2\ell-a} &\leq (\mathbf{E}X)^2 \frac{2\ell s}{n} O(1) n^{\ell/\ell_0} \\ &\leq (\mathbf{E}X)^2 O(1) \frac{\ell s}{n} n^{1/(4(r-1))} \\ &= (\mathbf{E}X)^2 O(1) \frac{\log^2 n}{n^{(4r-5)/(4r-4)}} = o((\mathbf{E}X)^2).\end{aligned}$$

8. Lower and upper bound for $1/(n^{r-1+1/\ell}) \ll p \ll 1/(n^{r-1+1/(\ell+1)})$ where $\ell = O(1)$.

8.1. **Upper bound.** Recall that from Section 6.2 we know that

$$\mathbf{E}X = \binom{n}{(\ell+1)(r-1)+1} p^{\ell+1} \leq n^{(\ell+1)(r-1)+1} p^{\ell+1} \ll n^{(\ell+1)(r-1)+1} \cdot 1/(n^{(\ell+1)(r-1)+1}) = 1$$

implying that $\mathbf{E}X$ tends to 0.

8.2. **Lower bound.** Similarly,

$$\mathbf{E}X = \binom{n}{\ell(r-1)+1} p^\ell \gg \left(\frac{n}{\ell(r-1)+1}\right)^{\ell(r-1)+1} \cdot 1/(n^{\ell(r-1)+1}) = \Omega(1)$$

yielding that $\mathbf{E}X$ tends to infinity.

We will now modify calculations from Section 5. Clearly, $\ell = O(1)$ and $s = O(1)$. Observe that in (2) we can use the following upper bounds,

$$2\left(\frac{s}{n-s}\right)^{r-1} p^{-1} \ll 1/n^{r-1} \cdot n^{r-1+1/\ell} = n^{1/\ell}.$$

Hence,

$$\begin{aligned}\sum_{1\leq a\leq \ell} \sum_{0\leq b\leq a-1} n_{a,b} p^{2\ell-a} &\leq (\mathbf{E}X)^2 \frac{2\ell s}{n} \sum_{1\leq a\leq \ell} \left(2\left(\frac{s}{n-s}\right)^{r-1} p^{-1}\right)^a \\ &\ll (\mathbf{E}X)^2 \frac{O(1)}{n} O(n) = (\mathbf{E}X)^2 O(1),\end{aligned}$$

showing that

$$\sum_{1\leq a\leq \ell} \sum_{0\leq b\leq a-1} n_{a,b} p^{2\ell-a} = o((\mathbf{E}X)^2).$$

8.3. **Method of moments.** Observe that if $p = c/(n^{r-1+1/\ell})$, then

$$\mathbf{E}X = \binom{n}{\ell(r-1)+1} p^\ell \sim \frac{n^{\ell(r-1)+1}}{(\ell(r-1)+1)!} \cdot c/(n^{\ell(r-1)+1}) = c/(\ell(r-1)+1)! =: \lambda.$$

Now by generalizing the second moment calculations one can show that

$$\mathbf{E}X(X-1)\cdots(X-k+1) \to \lambda^k$$

for any $k \geq 1$. Thus, the method of moments yields the statement (see, e.g. [3]).

9. Concluding remarks

We have found high probability estimates for $\ell_{\max}$ for all ranges of p. The gaps between upper and lower bounds are all of order $O(1)$ except for Theorem 1.5 where $\Omega(1/n^{(r-1)/2}) = p = O(\sqrt{\log n}/n^{(r-1)/4})$. Here we are off by an order $O((\log n)^{1/(r-1)})$ factor. It would be of some interest to remove this.

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