Complexity of a 3-dimensional assignment problem

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We show that a certain 3-dimensional assignment problem is NP-complete. To do this we show that the following problem is NP-complete: given bipartite graphs G_1, G_2 with the same sets of vertices, do there exist perfect matchings M_1, M_2 of G_1, G_2 respectively such that $M_1 \cap M_2 = \emptyset$?

1. Introduction

Let P, Q, R be 3 finite disjoint sets of equal size. For $u = (p, q, r) \in T = P \times Q \times R$ we define $s(u) = \{(p, q), (p, r), (q, r)\}$ and for $A \subseteq T$ we let $s(A) = \bigcup_{u \in A} s(u)$.

A set $A \subseteq T$ is called a *partial assignment* if u, $v \in A$ implies $s(u) \cap s(v) = \emptyset$.

A *total assignment* A is a partial assignment which satisfies $s(A) = (P \times Q) \cup (P \times R) \cup (Q \times R)$.

In this paper we prove the NP-completeness of the following *3-dimensional assignment* problem (3DA):

Instance: disjoint finite sets P, Q, R of equal size. A set $S \subseteq P \times Q \times R$.

Question: does there exist a total assignment $A \subseteq S$?

This is a special case of the integer programming problem (with $a_{ijk} = 0$ or 1)

$$\text{maximize} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} a_{ijk} x_{ijk},$$

subject to

$$\sum_{i=1}^{m} x_{ijk} = 1, \quad j, k = 1, \ldots, m,$$

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$$x_{ijk} = 0 \text{ or } 1, \quad i, j, k = 1, \ldots, m.$$

This has been studied by Burkard and Frohlich [1], Leue [6] and Vlach [7].

One application of 3DA is the following timetabling problem: given m teachers, m classes and m time periods and a set S of triples (p, q, r) where we can schedule teacher p to meet class q in time period r, we ask is it possible, using only triples from S, to arrange that in each time period, each teacher is assigned to exactly one class and vice-versa, so that after m periods every teacher has taught every class?

Note that this is a different timetabling problem to that discussed in Even, Itai and Shamir [2].

It is shown that 3DA is a special case of the 3-dimensional matching problem of Karp [5].

2. Complexity proof

Given disjoint finite sets P, Q of equal size, a *matching* is a set $M \subseteq P \times Q$ such that $|M| = |P|$ and every element of $P \cup Q$ occurs in exactly one ordered pair of M.

The NP-completeness of 3DA will be shown to follow from the NP-completeness of

DISJOINT MATCHINGS (DM)

Instance: disjoint finite sets P, Q with $|P| = |Q|$ and sets $A_1, A_2 \subseteq P \times Q$.

Question: do there exist matchings $M_i \subseteq A_i$ for $i = 1, 2$ such that $M_1 \cap M_2 = \emptyset$?

The known NP-complete problem which will be reduced to DM is MONOTONE ONE-IN-THREE SAT (1-3SAT) – Garey and Johnson [4, p. 259].

1-3SAT

Instance: A set $V = \{v(i): i = 1, \ldots, n\}$ of boolean variables. A set $C = \{C_1, \ldots, C_m\}$ of clauses such that (i) $|C_i| = 3$, $i =$

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$1,\dots,m$, (ii) no clause contains neglected variables.

Question: is there a truth assignment for V such that each clause in C has exactly one true variable?

Theorem 2.1. 1-3SAT $\propto$ DM.

Proof. Suppose that $C_i = \{v(k): k \in K_i\}$ where $|K_i| = 3$ for $i = 1,\dots,m$. We construct the following instance of DM:

$$P = X \cup D \cup F,$$
$$Q = Y \cup E \cup G$$

where

$$X = \{x[i,j]: i = 1,\dots,2m, j = 1,\dots,n\},$$
$$Y = \{y[i,j]: i = 1,\dots,2m, j = 1,\dots,n\},$$
$$D = \{d[i]: i = 1,\dots,m\},$$
$$E = \{e[i]: i = 1,\dots,m\},$$
$$F = \{f[i,k]: i = 1,\dots,m, k \in K_i\},$$
$$G = \{g[i,k]: i = 1,\dots,m, k \in K_i\}.$$

Definition of A_1. For $j = 1,\dots,n$ let

$$V[j] = \{(x[i,j], y[i,j]): i = 1,\dots,2m\}$$

and

$$\bar{V}[j] = \{(x[i,j], y[i+1,j]): i = 1,\dots,2m\}$$

$(y[2m+1,j] = y[1,j])$ then

$$A_1 = \bigcup_{j=1}^{n} (V[j] \cup \bar{V}[j]) \cup \{(d[i], e[i]): i = 1,\dots,m\} \cup \{(f[i,k], g[i,k]): i = 1,\dots,m, k \in K_i\}.$$

We note (see Fig. 1) that

if $M \subseteq A_1$ is a matching then for $j = 1,\dots,n$ we have

$$M \supseteq V[j] \text{ and } M \cap \bar{V}[j] = \emptyset \quad (\text{models } v(j) = \text{true}) \tag{2.1}$$

or

$$M \cap C[j] = \emptyset \text{ and } M \supseteq \bar{V}[j] \quad (\text{models } v(j) = \text{false}).$$

Definition of A_2. For $i = 1,\dots,m$ let

$$\mathrm{CS}[i] = \bigcup_{k \in K_i} \{(x[2i-1,k], y[2i-1,k]), (x[2i-1,k], y[2i,k]), (f[i,k], y[2i,k]), (f[i,k], e[i]), (d[i], y[2i-1,k]), (x[2i,k], g[i,k])\}$$

(see Fig. 2).

For a given j let

$$\{x[i,j]: j \notin K_t \text{ where } t = \lceil i/2 \rceil\} = \{x[i(r),j]: r = 1,\dots,s_j\}$$

(*defines* $i(1), i(2), \dots$) where $i(1) < i(2) < \cdots < i(s_j)$ and let

$$Z[j] = \{(x[i(r+1),j], y[i(r),j]): r = 1,\dots,s_j\}$$

where $i(s_j + 1) = i(1)$.

Note that $Z[j] \cap (V[j] \cup \bar{V}[j]) = \emptyset$ provided that

Assumption. No variable $v(j)$ occurs in exactly $m-1$ clauses.

We can make the above assumption because there is a polynomial time algorithm that solves all instances of 1-3SAT that do not satisfy the as-

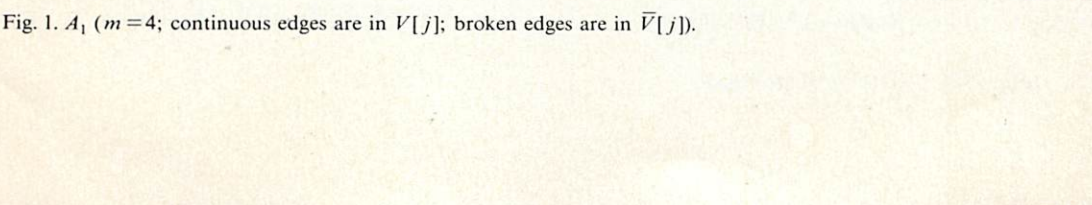

Fig. 1. A_1 ($m = 4$; continuous edges are in $V[j]$; broken edges are in $\bar{V}[j]$).

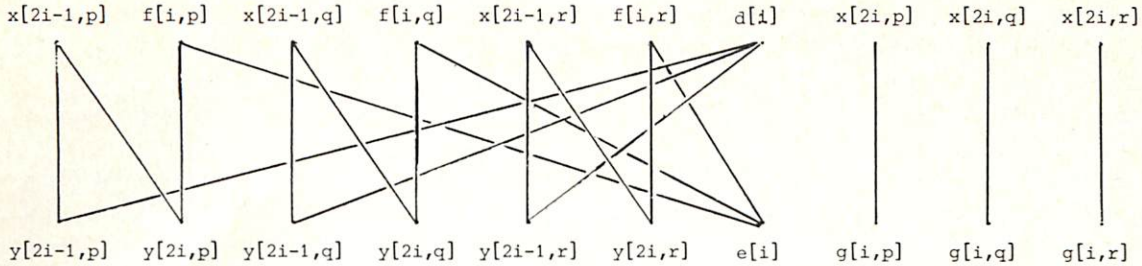

Fig. 2. CS[i] ($K_i = \{p, q, r\}$).

sumption.

Now let $Z = \bigcup_{j=1}^{n} Z[j]$ and let $A_2 = \bigcup_{j=1}^{m} \mathrm{CS}[i] \cup Z$. Note that $Z \cap A_1 = \emptyset$.

For $i = 1,\ldots,m$ and $k \in K_i$ define (see Fig. 3)

$$\begin{aligned}\mathrm{CM}[i,k] = \bigcup_{j\in K_i} &\{(x[2i,j], g[i,j])\}\\ &\cup \{(d[i], y[2i-1,k]),\\ &(x[2i-1,k], y[2i,k]), (f[i,k], e[i])\}\\ &\cup \bigcup_{j\in K_i-\{k\}} \{(x[2i-1,j], y[2i-1,j]),\\ &(f[i,j], y[2i,j])\}.\end{aligned}$$

We next note that

if $M \subseteq A_2$ is a matching then for $i = 1,\ldots,m$ there exists $k_i \in K_i$ such that $M \cap \mathrm{CS}[i] = \mathrm{CM}[i, k_i]$. (2.2)

Note also the following properties of CM[i,k]:

$$\mathrm{CM}[i,k] \cap V[k] = \emptyset, \tag{2.3a}$$
$$\mathrm{CM}[i,k] \cap \bar{V}[k] \neq \emptyset, \tag{2.3b}$$
$$\mathrm{CM}[i,k] \cap V[j] \neq \emptyset, j \in K_i - \{k\}, \tag{2.3c}$$
$$\mathrm{CM}[i,k] \cap \bar{V}[j] = \emptyset, j \in K_i - \{k\}, \tag{2.3d}$$
$$\mathrm{CM}[i,k] \cap (V[j] \cup \bar{V}[j]) = \emptyset, j \notin K_i. \tag{2.3e}$$

We must now show that 1-3SAT has a solution if and only if the above example of DM has a solution.

Suppose first that 1-3SAT has a satisfying assignment of truth values. In one such assignment let

$$T = \{j: v(j) = \text{true}\} \quad\text{and}\quad \bar{T} = \{1,\ldots,n\} - T.$$

For $i = 1,\ldots,m$ we can by assumption define k_i by $T \cap K_i = \{k_i\}$. Then let

$$\begin{aligned}M_1 = \bigcup_{j\in T} V[j] \cup \bigcup_{j\in\bar{T}} \bar{V}[j]\\ \cup \{(d[i], e[i]): i = 1,\ldots,m\}\\ \cup \{(f[i,k], g[i,k]): i = 1,\ldots,m, k \in K_i\}\end{aligned}$$

and

$$M_2 = \bigcup_{i=1}^{m} \mathrm{CM}[i, k_j] \cup Z.$$

That $M_1 \cap M_2 = \emptyset$ follows from (2.3) and $Z \cap A_1 = \emptyset$.

Conversely suppose we are given disjoint matchings $M_i \subseteq A_i$ for $i = 1, 2$. Let $T = \{j: V[j] \subseteq M_1\}$ and assign $v(j) = $ true for $j \in T$ and $v(j) = $ false for $j \in \bar{T}$.

Next (using (2.2)) let k_i be defined by $M_2 \cap$

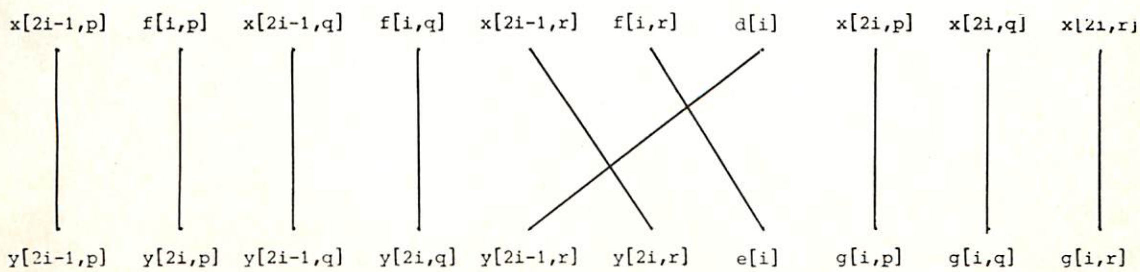

Fig. 3. CM[i, r] ($K_i = \{p, q, r\}$).

$CS[i]=CM[i,k_i]$ for $i=1,\ldots,m$. It follows from (2.3b) that $M_2 \cap \bar{V}[k_i] \neq \emptyset$. Thus if $M_1 \supseteq \bar{V}[k_i]$ we would have $M_1 \cap M_2 \neq \emptyset$. Thus $M_1 \not\supseteq \bar{V}[k_i]$ and so by (2.1) we have $M_1 \supseteq V[k_i]$ and $k_i \in T$ for $i=1,\ldots,m$. Now for $j \in K_i - \{k_i\}$ we use (2.3c) and (2.1) in a similar manner to show that $j \notin T$. Thus the truth value assignment satisfies all clauses in the required manner.

Corollary 2.2. 3DA *is* NP-*complete.*

Proof. We show that DM $\propto$ 3DA. Thus let $p=\{p_1,\ldots,p_m\}$, $Q=\{q_1,\ldots,q_m\}$, A_1, A_2 define an instance of DM. Let $R=\{1,\ldots,m\}$ and let $S=\bigcup_{i=1}^{m} S_i \subseteq P\times Q\times R$ where for $i=1$, 2, $S_i=A_i\times\{i\}$ and for $i=3,\ldots,m$, $S_i=P\times Q\times\{i\}$.

We need only note that any total assignment $A\subseteq S$ induces m disjoint matchings $M_i=\{(p,q): (p,q,i)\in A\}$ and that given a disjoint pair of matchings $M_i\subseteq A_i$ for $i=1$, 2 we can easily 'extend' them to a complete assignment. Indeed $(P\times Q)-(M_1\cup M_2)$ defines an $m-2$ regular bipartite graph which can be decomposed into $m-2$ disjoint matchings $M_3,\ldots,M_m$. Then $A=\bigcup_{i=1}^{m}(M_i\times\{i\})$ forms a total assignment and $A\subseteq S$.

We note next that 3DA is a special case of *3-dimensional matching* (3DM)

3DM

Instance: disjoint finite sets X, Y, Z of equal size. A set $T\subseteq X\times Y\times Z$.

Question: does there exist $B\subseteq T$ such that each element of $X\cup Y\cup Z$ occurs in exactly one member of B?

Remark 2.3. 3DA $\propto$ 3DM.

Given an instance P, Q, R, S of 3DA we proceed as follows: Let $X=P\times Q$, $Y=P\times R$, $Z=Q\times R$ and $T=\{((p,q),(p,r),(q,r)): (p,q,r)\in S\}$. It is clear that S contains a total assignment if and only if T contains a matching.

We finally note another hard special case of 3DM that can be deduced from 3DA: there exist $A\subseteq X\times Y$, $B\subseteq X\times Z$, $C\subseteq Y\times Z$ such that $T=\{(x,y,z): (x,y)\in A, (x,z)\in B \text{ and } (y,z)\in C\}$.

A practical instance of the above problem is described in Frieze and Yadegar [3].

This leads to an easy proof of NP-completeness of PARTITION INTO TRIANGLES (Garey and Johnson [4, pp. 68–69]) even when the graph under consideration is tripartite.

3. Complexity of DM

The instance of DM constructed in Theorem 2.1 has the following property: the bipartite graphs (P,Q,A_i) for $i=1$, 2 are both planar and no vertex has degree exceeding 3.

If we restrict the instance of DM to those with vertex degrees bounded by 2 then the problem becomes polynomially solvable even if we have to find disjoint matchings M_i of graphs (P,Q,A_i) for $i=1,\ldots,k$.

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