Distribution of vertex degree in web-graphs.

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Abstract

We give precise results on the age dependent distribution of degree, and degree sequence for two models of web-graphs (scale-free graphs) which we call the Undirected Process, and Hub-Authority Process. Our results for the Undirected Process generalize naturally to the equivalent hypergraph process.

1 Introduction

Models of processes variously called web-graphs or scale-free graphs have recently been studied by many authors. A recent survey by Bollobás and Riordan [3] gives an excellent overview of the topic and many detailed structural results.

A web-graph process generates a sequence of digraphs $G(t), t = 0, 1, ...$, where at step t, $G(t)$ is obtained from $G(t-1)$ in a manner which preserves the connectivity of the underlying graph. At step t, either a new vertex is added with edges joining it to $G(t-1)$ or extra edges are inserted within $G(t-1)$. Each edge inserted at step t independently chooses the required endpoint vertices from $G(t-1)$ according to a two step experiment. At the first step of the experiment, the method of choosing the endpoint is decided. The options are: either choose uniformly at random (u.a.r) or by copying (proportional to vertex degree). At the second step the endpoint vertex is sampled from $G(t-1)$. Web-graph processes require that there is a positive probability that copying occurs, and thus differ from classical random graph processes which use only uniform selection.

We consider two very distinct types of process giving rise to directed web-graphs, namely the *Undirected Process* and the *Hub-Authority Process*.

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Undirected Process. The copying rule chooses a vertex v for edge insertion according to total degree $d(v,t)$.

Hub-Authority Process. The copying rules choose vertices according to directed degree. Choice of initial vertex v for edge insertion is made according to out-degree $d^+(v,t)$. Choice of terminal vertex w for edge insertion is made according to in-degree $d^-(w,t)$. This method of copying mimics the *Hub-Authority* view of the WWW, proposed by Kleinberg in [11].

The number of vertices of the process at any step t is not fixed. We consider the *(proportional) degree sequence*, $N_t(k)$ which is the proportion of vertices of degree k in the process at step t. Clearly this is a random variable[1] and a function of t. It has two interesting properties which have been widely studied by many authors (see eg: [3] for many citations) namely that its is asymptotically independent of t (scale free), and exhibits a power law relationship, $N(k) \sim Ck^{-x}$, as $k \to \infty$ slowly. The value of x is a constant ($x > 2$) depending only on the copying aspects of the process.

The main results of this paper, which we now summarize informally, are given in Theorem 1 for the Undirected Process, and in Theorem 8 for the Hub-Authority Process. These theorems give precise values for the proportional degree sequence of the process. Thus Theorem 1 refines the results of [7], whereas the results of Theorem 8 are new.

In the Undirected process, for a fixed vertex v of initial degree m, the distribution of degree $d(v,t)$, of v at step t, as viewed over all processes $G(t)$, is

$$\mathbf{Pr}(d(v,t) = m + l \mid d(v,v) = m) \sim \binom{l + m + \frac{\nu}{\eta} - 1}{l} \left(\frac{v}{t}\right)^{\eta m + \nu} \left(1 - \left(\frac{v}{t}\right)^{\eta}\right)^{l}. \tag{1}$$

The parameters η and ν are given by (4), (5), and depend respectively on the uar and copying aspects of the process. By adding over v we obtain the expected number of vertices of degree $m + l$. However this variable is concentrated and an estimate for $N_t(m + l \mid m)$ follows. Let $n(l \mid m)$ denote the expected value of $N_t(l + m \mid m)$, then $n(l \mid m)$ is a ratio of Gamma functions, which approximate to a power law as $l \to \infty$ ie: $n(l \mid m) \sim Cl^{-x}$. The parameter x is given by $x = 1 + 1/\eta$ where $0 < \eta < 1$ is given by (4).

The distribution for $d(v,t)$ given above is valid for vertices born after time $v_0 = o(t)$ given by (8). This places an arbitrary upper bound $l_0 = o(t^{1/4})$ on the validity of the power law relationship for $n(l \mid m)$. The distribution (1) is derived in Theorem 2 on the assumption that $(1 + x)^m \sim 1 + mx$. Correct treatment of the higher order terms will presumably give the appropriate degree distribution of the early vertices.

It clear that the power law relationship for $n(l \mid m)$ cannot be correct for older vertices as it predicts a maximum degree of $\Theta(t^{\eta/(1+\eta)})$ whereas the correct maximum degree is $\Theta(t^{\eta})$. Interestingly, the experimental evidence of [4] for the distribution of vertex degree in the WWW has a much heavier tail than is compatible with a power law.

[1] Assertions made in this paper are in probability. Usually we qualify this explicitly with the abbreviation **whp** (with probability tending to 1 as $t \to \infty$).

For the Hub-Authority process, the picture is similar. The distribution of in-degree ($d^-(v,t) = r$) and out-degree ($d^+(v,t) = s$) of a given vertex v are more or less independent and are given by the product of two distributions of the type given in (1) above. By fixing the degrees r, s, and summing over v we obtain $n(r,s)$ the expected proportion of vertices of in-degree r and out-degree s. Typically the power law for $n(r,s)$ is of the form

$$n(r, s \mid m^-, m^+) = C_{r,s} r^{-x^-} s^{-x^+}.$$

The precise values of x^-, x^+ are given by (28), (29). We assume that at least one of $r, s \to \infty$. The values of the power law parameters depend on the relative sizes of r, s and change when s increases so that $s = \Theta(r^{\eta^+/\eta^-})$, where η is the copying parameter.

It is worth remarking that the results for the Hub-Authority process (Theorem 8) are obtained as a trivial extension of the proof for the Undirected process (Theorem 1). This proof is itself easier than deriving the degree sequence of the process using recurrence relations (for example, as in [7]). Deriving the degree sequence of the Hub-Authority process from recurrence relations seems rather difficult in the general case, although progress using this method has recently been made in [1].

The proof method used in this paper easily allows some natural generalizations of the web-graph process, which we mention briefly: the Undirected web-process for hypergraphs, which is covered in Section 6 and, allowing a general distribution on edge directions of new vertices in the Hub-Authority Process, see Section 5.3.

It is also possible to analyze a general directed model which copies based on a mixture of in- and out-degree (see [6]). Finally we note that it is possible to allow for edge deletion in the process, and the component structure of such Undirected processes is studied in [8].

The layout of this paper is as follows: In Section 2 we give a precise definition of the processes. In Section 3 we give a formal statement of results for the Undirected Process. In Section 4 we prove these results. In Section 5 we give results for the Hub-Authority Process. Our proof for the Undirected Process generalizes easily to the Hub-Authority Process, and we sketch an outline of any minor differences in the proof. We also give results for the generalized Hub-Authority process which is less restrictive on the distribution of edge directions of new vertices. As previously mentioned, the Undirected Process generalizes naturally to a hypergraph process. A brief summary of this is given in Section 6. Finally, in Section 7 we discuss a notion of equivalence between models of web-graphs (or scale-free graphs) based on a simple amortized process.

2 Definition of web-graph process

2.1 Undirected Process.

There is an initial connected (di)-graph $G(0)$ of finite size. At each step $t = 1, 2, ...,$ a choice is made between an OLD and a NEW procedure to be followed, according to an independent Bernoulii random variable. Let α be the probability that the NEW procedure is followed at step t, and $\beta = 1 - \alpha$ be the probability of the OLD procedure being followed.

NEW **procedure.** If the NEW procedure is chosen at step t, a new vertex v_t is added with m edges directed to $G(t-1)$. The number of edges m is sampled from a *finite* distribution $\boldsymbol{p}$. That is to say, $\boldsymbol{p} = (p_0, p_1, ..., p_m, ..., p_L)$, where $L < \infty$ and $p_m \geq 0$ constant. Let $\overline{m} = \mathbf{E}\, m$. Usually we require $p_0 = 0$, so that the web-graph is connected. For convenience we refer to vertex v_t by its step label t.

Each new edge $e_i, i = 1, ..., m$ makes an independent decision to select its terminal vertex w in $G(t-1)$ by copying or uniformly with probabilities $A_1,\ A_2 = 1 - A_1$ respectively. Thus

$$\mathbf{Pr}(w \text{ is selected by } e_i) = p_A(w,t) = A_1 \frac{d(w,t-1)}{d(t-1)} + A_2 \frac{1}{n(t-1)}, \tag{2}$$

where $d(t)$ is the degree of $G(t)$ and $n(t)$ is the number of vertices of $G(t)$.

OLD **procedure.** If the OLD procedure is followed then M random edges are inserted into $G(t-1)$ according to $\boldsymbol{q} = (q_1, ..., q_M, ..., q_{L'})$. Let $\overline{M} = \mathbf{E}\, M$.

The choice of initial vertex (resp. terminal vertex) of each inserted edge is made independently from uar/copy rules similar to (2), but with parameters B_1, B_2 (resp C_1, C_2). Thus following (2) let

$$p_Z(v,t) = Z_1 \frac{d(v,t-1)}{d(t)} + Z_2 \frac{1}{n(t)} \qquad \text{for } Z = A, B, C. \tag{3}$$

We remark that this model differs from the one described in [7], as the inserted OLD edges are not required to have a single initial vertex.

It will be shown that the degree distribution of the undirected process depends on two parameters $\eta,\ \nu$, given by

$$\eta = \frac{\alpha \overline{m} A_1 + \beta \overline{M}(B_1 + C_1)}{2(\alpha \overline{m} + \beta \overline{M})} \tag{4}$$

$$\nu = \frac{\alpha \overline{m} A_2 + \beta \overline{M}(B_2 + C_2)}{\alpha}. \tag{5}$$

2.2 Hub-Authority Process

The description is similar to the undirected case. Independently at each step either a NEW-IN procedure is followed with probability α, a NEW-OUT procedure with probability γ or an OLD procedure with probability β.

NEW-IN **procedure.** When this procedure is chosen (with probability α) at step t, a new vertex $v_t = t$ directs $m^-(t)$ edges from itself to $G(t-1)$. The distribution $\boldsymbol{p}^-$ of $m^-(t)$ is finite.

The terminal vertices for the new edges are selected by a mixture of uar and copying according to in-degree. Thus each new edge independently selects vertex w as a terminal vertex with probability

$$p_A(w,t) = A_1 \frac{d^-(w,t-1)}{d^-(t)} + A_2 \frac{1}{n(t)}. \tag{6}$$

NEW-OUT **procedure.** The edges are directed independently from $G(t-1)$ towards the new vertex $v_t = t$. The distribution of $\boldsymbol{p}^+$ is finite. The initial vertex of an edge is selected according to a mixture of uar and copying on out-degree. Thus each new edge independently selects vertex w as an initial vertex with probability

$$p_D(w,t) = D_1 \frac{d^+(w,t-1)}{d^+(t)} + D_2 \frac{1}{n(t)}. \tag{7}$$

OLD **procedure.** If this is followed, the distribution $\boldsymbol{q}$ of the number of edges is finite. Each edge makes an independent choice of initial and terminal vertex. The initial vertex of the edge is chosen according to an out-degree/uar mixture with probability $p_B(w,t)$ similar in definition to $p_D(w,t)$ in (7). The terminal vertex is chosen according to an in-degree/uar mixture with probability $p_C(w,t)$, similar in definition to p_A in (6).

3 The Undirected Process: Statement of results

It is convenient to classify vertices according their time of birth relative to the current step t. We say an *early* vertex, is a vertex born before $v_0(t) = o(t)$, where the precise value of $v_0(t)$ is given in (8) below. A *late* vertex, is one born after this time. Let

$$v_0 = \lceil \left(t^{\frac{4\eta}{4\eta+1}} \right) \log^5 t \rceil \tag{8}$$

The results we give are as follows: For any late vertex we give the expectation and time dependent distribution of the current degree, $d(v,t)$. We also give precise and asymptotic values of the degree sequence. For early vertices we give bounds for the expected maximum degree of the process.

For presentation of asymptotic results we use $a(t) \sim b(t)$ to mean $\lim_{t\to\infty} a(t)/b(t) = 1$. In a different context, we also use $X \sim D$ to denote a random variable X with distribution D. We adopt the usual definition of the generalized binomial coefficient, namely for real $a, b \geq 0$,

$$\binom{a+b}{a} = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\,\Gamma(b+1)}. \tag{9}$$

We take an indirect approach in which we derive the distribution of degree $d(v,t)$ of a fixed vertex v over all processes which include this vertex. We then use concentration of degree sequence to infer the structure of a typical process.

Theorem 1. *Let $\xi(m)$ be given by*

$$\xi = m + \nu/\eta, \tag{10}$$

where η, ν are given by (4), (5).

(a) Distribution of degree of late vertices is approximated by a negative binomial distribution. More precisely, for $v \geq v_0$ and any $K > 0$ constant,

$$\mathbf{Pr}(d(v,t) = \ m + l \mid d(v,v) = m) = (1+o(1))\binom{l+\xi-1}{l}\left(\frac{v}{t}\right)^{\eta m+\nu}\left(1-\left(\frac{v}{t}\right)^{\eta}\right)^{l}.$$

Upper tail of degree distribution of late vertices:

$$\mathbf{Pr}\left(\exists[v \geq v_0, t \geq v] : d(v,t) \geq K \log t \left(\frac{t}{v}\right)^{\eta}\right) = o({v_0}^{-K/4}) \qquad K \geq 8.$$

(b) Expected degree of late vertices.

$$\mathbf{E}\ (d(v,t) \mid d(v,v) = m) \sim m + \left(m + \frac{\nu}{\eta}\right)\left(\left(\frac{t}{v}\right)^{\eta} - 1\right).$$

(c) Distribution of late vertices.
Provided $t/v \to \infty$, then under the transformation $l = x(t/v)^{\eta}$, this tends to a Gamma distribution $g(x)$ with pdf given by

$$g(x) = \frac{x^{m+\nu/\eta-1}}{\Gamma(m+\nu/\eta)} e^{-x}.$$

(c) Proportional degree sequence.
Let

$$\widehat{n}(l,m) = \frac{p_m}{\eta}\frac{\Gamma(m+(\nu+1)/\eta)}{\Gamma(m+\nu/\eta)}\frac{\Gamma(l+m+\nu/\eta)}{\Gamma(l+1+m+(\nu+1)/\eta)}.$$

For $l = o(t^{\eta/(1+\eta)})$, let $n_t(l,m)$ denote the expected proportion of vertices of $G(t)$ of degree $l+m$ with initial degree m. Then **whp** *$n_t(l,m) \sim \widehat{n}(l,m)$. In particular, as $l \to \infty$, there is a Power Law asymptotic given by*

$$n(l,m) \sim \tfrac{p_m}{\eta}\tfrac{\Gamma(m+(\nu+1)/\eta)}{\Gamma(m+\nu/\eta)}\, l^{-\left(1+\frac{1}{\eta}\right)}.$$

(d) *Concentration.*
Let $l = o(t^{1/4})$. Let $D_t(l,m) = |\{v : d(v,t) = l, d(v,v) = m\}|$, and let $\mu_{l,m} = \mathbf{E}\, D_t(l,m)$ then

$$\mathbf{Pr}\left(|D_t(l,m) - \mu_{l,m}| \geq \frac{\mu_{l,m}}{\log t}\right) = O\left(l^{5/2+1/2\eta}\frac{\log^3 t}{\sqrt{t}}\right).$$

(e) *Expected maximum degree.*
Let $\Delta(G(t))$ denote the maximum degree of $G(t)$ then

$$\mathbf{E}\,\Delta(G(t)) = \Theta(1)t^{\eta}.$$

To reassure the sceptical reader that indeed $\sum_{m,l} n(l,m) = 1$ we note that $n(l,m)$ can be written as

$$n(l,m) = \frac{p_m}{\eta}\frac{\Gamma(\xi+1/\eta)}{\Gamma(\xi)\Gamma(1+1/\eta)}B(l+\xi, 1+1/\eta),$$

where $B(a,b) = \int x^{a-1}(1-x)^{b-1}\,dx$ is the Beta Integral. Now

$$\sum_{l\geq 0} B(l+\xi, 1+1/\eta) = B(\xi, 1/\eta).$$

Hence $\sum_l n(l,m) = p_m$ and the consistency of the above claim follows.

An interesting paper [5] by Buckley and Osthus was recently brought to our attention, and requires special mention. The results of [5], which are for the initial-attractiveness variant of the LCD model, are similar to those stated above. They are however restricted to a model which follows a NEW procedure with r edges at each step and with $A_1 = 1/(a+1)$, $A_2 = a/(a+1)$, where a is a natural number. The analysis is made for the distribution of in-degree of any vertex, which can always be done for the Undirected Process when the out-degree (r) is fixed. This leads to the removal of a factor of $\frac{1}{2}$ in the definition of η in (4). See [7] for details. In the notation of this paper, this is a (modified) Undirected Process with $\alpha = 1$, and edge distribution $p_r = 1$. Thus $\nu = 0$, $\eta = 1/(a+1)$ and $\xi = r$.

4 Proof of Theorem 1: The Undirected Process

Let $d(t)$ (resp. $n(t)$) be the total degree (resp. number of vertices) of $G(t)$. Thus

$$\begin{aligned}\mathbf{E}\,d(t) =& 2t(\alpha\overline{m} + (1-\alpha)\overline{M}) + d(0),\\ \mathbf{E}\,n(t) =& t\alpha + n(0).\end{aligned}$$

The values of $d(t)$, $n(t)$ are the sum of independent bounded random variables and are sharply concentrated in t. The inequality

$$h(t) = \mathbf{E}\,h(t)\left(1+O\left(\sqrt{\tfrac{K(1+\log t)}{t}}\right)\right) \qquad h(t) = d(t),\ n(t) \qquad t \geq 1 \tag{11}$$

holds simultaneously with probability $1 - {t_0}^{-\alpha K+1}$ for all $t \geq t_0$ and $t_0 > 0$. Let $\boldsymbol{C}(t)$ denote the set of processes $G(t)$ for which (11) holds. We refer to such processes as *good*.

4.1 Time dependent degree distribution

Theorem 2. *Let η, ν, ξ be given by (4), (5), (10), and let v_0 be given by (8). Let $l = o((t/\log t)^{1/4})$. Let $\xi(m) = m + \nu/\eta$, and let*

$$\pi_{l,m}(v,t) = \binom{l+\xi-1}{l}\left(\frac{v}{t}\right)^{\eta\xi}\left(1-\left(\frac{v}{t}\right)^{\eta}\right)^{l}.$$

Let $\delta^*(l,v)$ be given by

$$\delta^* = O\left((m+l)\sqrt{\frac{\log v}{v}} + \frac{(m+l)^2}{v}\right). \tag{12}$$

Let $v, w \in \{1, ..., t\}$, $v, w \geq v_0$. Suppose that, at step v a new vertex of initial degree m is added, and at step w a new vertex of degree m' is added, then:

(a)

$$\mathbf{Pr}(d(v,t) = \; m + l \mid d(v,v) = m) = \pi_{l,m}(v,t)\left(1 + l\delta^*(l,v)\right). \tag{13}$$

(b)

$$\begin{aligned}\mathbf{Pr}(d(v,t) = m+l,\ d(w,t) = m'+l' \mid d(v,v) = m,\ d(w,w) = m') \\ = \pi_{l,m}(v,t)\pi_{l',m'}(w,t)\left(1 + (l+l')(\delta^*(l+l',v) + \delta^*(l+l',w))\right).\end{aligned} \tag{14}$$

Proof We first give an outline description of the proof of (a), and then fill in the details.

Let $\mu_t = (1_\alpha(t), m(t), M(t))$ give the choice of procedure (NEW or OLD), and number of edges inserted at step t, and let $\boldsymbol{\mu}(t) = (\mu_1, ..., \mu_t)$ be the history of these choices. Let $\boldsymbol{G}(\boldsymbol{\mu}(t)) = \{G(t) : \boldsymbol{\mu}(t)\}$. All graphs in $\boldsymbol{G}(\boldsymbol{\mu}(t))$ have the same degree $d(T)$ and number of vertices $n(T)$, for all $T \leq t$.

Let $\boldsymbol{\mu}(t)$ be such that $\mu_v = (1, m, 0)$. Thus a new vertex labeled v, of initial degree m is added at step v. Let $d(v,t)$ denote the degree of the vertex v at the end of step t.

Suppose $d(v,t) = m + l$ and that $\boldsymbol{\tau} = (\tau_1, ..., \tau_l)$ gives the steps τ_j at which $d(v, \tau_j)$ changes from $m + j - 1$ to $m + j$. If more than one edge hits v during step τ_j the value τ_j is repeated the appropriate number of times in $\boldsymbol{\tau}$. We define $\tau_0 = v$ and $\tau_{l+1} = t$.

The set $\boldsymbol{C}(t)$ of good sequences is defined in (11) above. Let $I(t)$ be given by

$$I(t) = \{\boldsymbol{\mu}(t) \in \boldsymbol{C}(t) : 1_\alpha(v) = 1,\ d(v,v) = m\}.$$

By working within $I(t)$ we can replace $d(T), n(T)$ in (3) by their approximate expected values to remove conditioning on previous history of these variables at each step $T = v+1, ..., t$. The approximation error we incur at each step is $(1 + O(\sqrt{\log T/T}))$.

The probability $\mathbf{Pr}(d(v,t)=m+l,\ \boldsymbol{\tau}\mid d(v,v)=m)$ is obtained as follows:

$$\mathbf{Pr}(d(v,t)=m+l,\ \boldsymbol{\tau}\mid d(v,v)=m)=\prod_{j=1,\dots,l+1}\mathbf{Pr}(\tau_j\mid\tau_{j-1},\ d(v,v)=m).$$

Here, $\mathbf{Pr}(\tau_{l+1}\mid\tau_l)$ is taken to mean that the degree of v remains unaltered at $m+l$ from step τ_l up to and including $\tau_{l+1}=t$. Finally we sum $F(\boldsymbol{\tau})$ over $\boldsymbol{\tau}$ to get $\mathbf{Pr}(d(v,t)=m+l\mid d(v,v)=m)$. We now give the detailed analysis.

Let $X(t)$ count the number of hits on vertex v at step t. The distribution of $X(t)$ is given by

$$X(t)\sim 1_\alpha(t)\mathrm{Bin}(m(t),p_A(t))+1_\beta(t)[\mathrm{Bin}(M(t),p_B(t))+\mathrm{Bin}(M(t),p_C(t))],$$

where $\mathrm{Bin}(m,p)$ is the Binomial distribution, and the parameters are described in Section 2.1. Conditional on $d(v,t)=m+j$ and $I(t)$, and recalling the definitions (3), (4) and (5) we have

$$\mathbf{E}\,X(t+1\mid d(v,t)=m+j,I(t))=\frac{\eta(m+j)+\nu}{t}\left(1+O\left(\sqrt{\tfrac{\log t}{t}}\right)\right).$$

Let

$$\epsilon(t,j)=O(\sqrt{\log t}/t+(m+j)/t),\tag{15}$$

then

$$\begin{aligned}\mathbf{Pr}(X(t+1)=&0\mid d(v,t)=m+j,I(t)) &(16)\\ =&\mathbf{E}_{\ \mu_{t+1}}\left(1_\alpha(1-p_A)^{m(t+1)}+1_\beta[(1-p_B)^{M(t+1)}(1-p_C)^{M(t+1)}]\right) &(17)\\ =&1-\left(\frac{\eta(m+j)+\nu}{t}\left(1+\epsilon(t,j)\right)\right). &(18)\end{aligned}$$

If $\tau_{j+1}=\tau_j$ or τ_j+1 let $\Psi_j=1$. If $\tau_{j+1}\geq\tau_j+2$, let $\Psi_j=\mathbf{Pr}(X(T)=0,\tau_j<T<\tau_{j+1})$, thus

$$\Psi_j=\prod_{\tau_j<T<\tau_{j+1}}1-\left(\frac{\eta(m+j)+\nu}{T}\left(1+\epsilon(T,j)\right)\right).\tag{19}$$

Let $\delta(t,j)=O((m+j)\sqrt{\log t/t}+(m+j)^2/t)$, then

$$\begin{aligned}\Psi_j&=\exp\left(-(\eta(m+j)+\nu)\sum_{\tau_j<T<\tau_{j+1}}\frac{1}{T}+\delta(\tau_j,j)\right)\\ &=\left(\frac{\tau_j}{\tau_{j+1}}\right)^{\eta(m+j)+\nu}(1+\delta(\tau_j,j)).\end{aligned}$$

Let $\Phi_l=1$ and for $0\leq j\leq l-1$, let $\Phi_j=\mathbf{Pr}(X(t+1)=1\mid d(v,t)=m+j,I(t))$. Thus

$$\Phi_j=\frac{\eta(m+j)+\nu}{t}\left(1+\epsilon(t,j)\right).$$

Also, $\mathbf{Pr}(X(t+1)\geq 2)=O(\Phi_j\ (m+j)/t)$. Thus, when the entries in $\boldsymbol{\tau}$ are not distinct, the correction arising from these events can be absorbed into $\delta(\tau_j,j)$.

Let $F(\boldsymbol{\tau})$ denote $\mathbf{Pr}(d(v,t)=m+l$ and $\boldsymbol{\tau}\mid d(v,v)=m,I(t))$ then

$$F(\boldsymbol{\tau})=\prod_{j=1,\dots,l+1}\mathbf{Pr}(\tau_j\mid\tau_{j-1},\ d(v,v)=m,I(t))=\prod_{j=0}^{l}\left(\Psi_j\ \Phi_j\right).$$

Ignoring for the moment the multiplicative error terms, we see that

$$F(\boldsymbol{\tau})\sim\left(\frac{v}{\tau_1}\right)^{\eta m+\nu}\frac{\eta m+\nu}{\tau_1}\left(\frac{\tau_1}{\tau_2}\right)^{\eta(m+1)+\nu}\frac{\eta(m+1)+\nu}{\tau_2}\cdots\left(\frac{\tau_{l-1}}{\tau_l}\right)^{\eta(m+l-1)+\nu}\frac{\eta(m+l-1)+\nu}{\tau_l}\left(\frac{\tau_l}{t}\right)^{\eta(m+l)+\nu}.$$

Cancelling repeated values of τ_j from the numerator and denominator of $F(\boldsymbol{\tau})$ gives

$$F(\boldsymbol{\tau})=\binom{l+\xi-1}{l}\left(\frac{v}{t}\right)^{\eta m+\nu}\times l!\prod_{j=1}^{l}\frac{\eta\ \tau_j^{\eta-1}}{t^\eta}\left(1+\delta(\tau_j,j)\right), \tag{20}$$

where $\xi=m+\nu/\eta$. Thus

$$\mathbf{Pr}(d(v,t)=m+l\mid d(v,v)=m)=\left(1+O\left(\frac{1}{t^K}\right)\right)\sum_{\boldsymbol{\tau}}F(\boldsymbol{\tau})+O\left(\frac{1}{t^K}\right).$$

Now

$$\begin{aligned}l!\sum_{\boldsymbol{\tau}}\prod_{j=1}^{l}\frac{\eta\tau_j^{\eta-1}}{t^\eta}(1+\delta(\tau_j,j)) &= \left(\sum_{v<\tau\leq t}\frac{\eta\tau^{\eta-1}}{t^\eta}(1+\delta(\tau,l))\right)^l\\ &= \left(1-\left(\frac{v}{t}\right)^\eta(1+\delta^*)\right)^l,\end{aligned}$$

where

$$\delta^*=O\left((m+l)\sqrt{\frac{\log v}{v}}+\frac{(m+l)^2}{v}+(m+l)\sqrt{\frac{\log t}{t}}\left(\frac{t}{v}\right)^\eta+\frac{(m+l)^2}{t}\left(\frac{t}{v}\right)^\eta\right). \tag{21}$$

Thus

$$\mathbf{Pr}(d(v,t)=m+l\mid d(v,v)=m)=\pi_{l,m}(v,t)(1+l\delta^*),$$

as required.

This gives the value of (13) and thus completes the proof of part (a) of the theorem.

The proof of part (b) of the theorem is similar to that of part (a). We give a reasonably complete exposition which can be easily imitated for use in the Hub-Authority process. The proof works without alteration because, given $d(v,t)=m+j,\ d(w,t)=m'+j'$ then

$$p_A(v,t)=O\left(\frac{m+j}{t}\right)=O(\epsilon(t,j+j')),$$

where $\epsilon(t, j) = O(\sqrt{\log t}/t + (m + j)/t)$ given by (15) is the one-step approximation error in the derivation of part (a).

Let $v < w$ and let $\boldsymbol{\sigma}(v), \boldsymbol{\sigma}(w)$ be the transition sequences for v, w, and let $\boldsymbol{\tau}$ be the sequence obtained by merging $\boldsymbol{\sigma}(v), \sigma(w)$. The random variables $X_v(t), X_w(t)$ for the number of times the vertices v, w are chosen at step t, now have the following joint probabilities:

$$\begin{aligned}\mathbf{Pr}(X_v(t) = 0, &X_w(t) = 0 \mid d(v, t-1) = m + j, d(w, t-1) = m' + j', I_{v,w}(t)) \\ &= 1 - \left(\frac{\eta(m + j) + \nu + \eta(m' + j') + \nu}{t} \left(1+\epsilon(t,j+j')\right)\right).\end{aligned}$$

$$\begin{aligned}\mathbf{Pr}(X_v(t) = 1, &X_w(t) = 0 \mid d(v, t-1) = m + j, d(w, t-1) = m' + j', I_{v,w}(t)) \\ &= \frac{\eta(m + j) + \nu}{t} \left(1(t < w) + 1(t \geq w)(1 - p_A(w, t))^{m(t)}\right) \\ &= \frac{\eta(m + j) + \nu}{t}(1 + \epsilon(t, j + j')).\end{aligned}$$

Finally

$$\mathbf{Pr}(X_v(t) \geq 1, X_w(t) \geq 1) = \Theta(\mathbf{Pr}(X_v(t) \geq 2)),$$

which is the correct order of error used in part (a).

The joint distribution of $d(v, t)$ and $d(w, t)$ derives itself rather naturally as follows:

Let $n(j, v)$ be the number of occurrences of $\boldsymbol{\sigma}(v)$ in $(\tau_1, ..., \tau_j)$, then following (19)

$$\Psi_j = \left(\frac{\tau_j}{\tau_{j+1}}\right)^{\eta(m+n(j,v))+\nu+\eta(m'+n(j,w))+\nu \left(1+\delta(\tau_j,n(j,v)+n(j,w))\right)},$$

and

$$\Phi_j = \frac{\lambda_j}{\tau_j}\left(1 + \epsilon(\tau_j, n(j, v) + n(j, w))\right),$$

where

$$\lambda_j = \begin{cases} \eta(m + n(j, v)) + \nu & \text{if } \tau_j \in \boldsymbol{\sigma}(v) \\ \eta(m' + n(j, w)) + \nu & \text{if } \tau_j \in \boldsymbol{\sigma}(w) \end{cases}$$

Thus $F(\boldsymbol{\tau}) = H(\boldsymbol{\sigma}(v))H(\boldsymbol{\sigma}(w))$ where $F(\boldsymbol{\tau})$ is given by (20), and

$$\begin{aligned} H(\boldsymbol{\sigma}(v)) &= \left(\prod_{j\in\boldsymbol{\sigma}(v)} \frac{\lambda_j}{\tau_j}\right) \left(\prod_{j=0}^{l+l'} \left(\frac{\tau_j}{\tau_{j+1}}\right)^{\eta(m+n(j,v))+\nu \left(1 + \delta(\tau_j, n(j, v) + n(j, w))\right)}\right) \\ &= \left(\prod_{i=0}^{l-1} \frac{\eta(m + i) + \nu}{\sigma_i}\right) \left(\prod_{i=0}^{l} \left(\frac{\sigma_i}{\sigma_{i+1}}\right)^{\eta(m+i)+\nu \left(1 + \delta(\sigma_i, i + l')\right)}\right) \\ &= F(\boldsymbol{\sigma}(v)), \end{aligned}$$

as required. Thus $F(\boldsymbol{\tau}) = F(\boldsymbol{\sigma}(v))F(\boldsymbol{\sigma}(w))$ which completes the proof of part (b). $\square$

4.2 Results deriving from Theorem 2.

We first prove an almost sure maximum degree bound for late vertices.

Lemma 1.

$$\mathbf{Pr}\left(\exists\, [v \geq v_0,\ t \geq v] \textit{ such that } d(v,t) \geq (2K+4)(\log t)(t/v)^{\eta} \right) = o({v_0}^{-K})$$

Proof

Given v, and choosing $l(v) = o((v/\log v)^{1/4})$, we see that for $l \leq l(v)$ we have $l\delta^* = o(1)$ in (13), where δ^* is given by (12). Thus, if $l = K(\log t)(t/v)^{\eta}$ then provided $v \geq v_0 = (\log^5 t)t^{4\eta/(1+4\eta)}$ we have $l \leq l(v)$.

Noting that

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}, \tag{22}$$

and recalling (9), then as $l \to \infty$,

$$\binom{l+m+\nu/\eta-1}{l} \sim \frac{l^{m+(\nu/\eta)-1}}{\Gamma(m+(\nu/\eta))}. \tag{23}$$

Hence,

$$\mathbf{Pr}(d(v,t) = m+l) \leq O(1)\, l^{m+(\nu/\eta)-1} \left(\frac{v}{t}\right)^{\eta m+\nu} e^{-l\left(\frac{v}{t}\right)^{\eta}}.$$

Let $\lambda(v,t) = \lfloor K \log t (t/v)^{\eta} \rfloor$, then if $t \to \infty$

$$\begin{aligned} \mathbf{Pr}(d(v,t) \in [m+\lambda, m+2\lambda]) &\leq (\log t)^{m+(\nu/\eta)} e^{-K(1-o(1))\log t} \\ &= o(t^{-K/2}). \end{aligned}$$

As the edge distributions $\boldsymbol{p}, \boldsymbol{q}$ are finite, at most $M = \max(L, L') < \infty$ edges can be added to v at each step. Provided $d(v,v) = m \ll K \log v$ and $\lambda(v,v) = K \log v \to \infty$, which it does if $v \to \infty$, it is impossible to jump across the $[\lambda, 2\lambda]$ gap at any step. Thus, we have inductively

$$\mathbf{Pr}(\exists t \geq v : d(v,t) \geq \lambda(v,t)) = o(v^{-K/2+1}),$$

and the result follows. □

This completes the proof of Theorem 1(a). The proof of Theorem 1(b) now follows because the negative binomial distribution NBin(r,p) has expected value $r(1-p)/p$.

Corollary 3.

$$\mathbf{E}\ (d(v,t) \mid d(v,v) = m) \sim m + (m+(\nu/\eta)) \left(\left(\frac{t}{v}\right)^{\eta} - 1 \right).$$

The following result, for the asymptotic distribution of the number of vertices of degree l, gives Theorem 1(c):

Corollary 4. *Let $(t/v) \to \infty$. Under the transformation $l = x(t/v)^\eta$ the probability $\mathbf{Pr}(d(v,t) = m+l \mid d(v,v) = m)$ given by (13) tends to a Gamma distribution with pdf $g(x)dx$ given by*

$$g(x) = \frac{x^{m+\nu/\eta-1}}{\Gamma(m+\nu/\eta)} e^{-x}.$$

Proof This follows from inserting (23) into (13). □

Corollary 5. *Let*

$$n(l,m) = \frac{p_m}{\eta} \frac{\Gamma(l+m+\nu/\eta)\,\Gamma(m+(\nu+1)/\eta)}{\Gamma(m+\nu/\eta)\,\Gamma(m+l+1+(\nu+1)/\eta)}. \tag{24}$$

Let $N_t(l,m)$ denote the proportion of vertices of $G(t)$ of degree $l+m$ which had initial degree m then $\mathbf{E}\ N_t(l,m)) \sim n(l,m)$.

Proof

Let $D_t(l,m)$ be the number of vertices v with initial degree m and $d(v,t) = l+m$. Then

$$\begin{aligned}
\mathbf{E}\ D_t(l,m) &= O(v_0) + \sum_{v \geq v_0} \alpha p_m \mathbf{Pr}(d(v,t) = m+l \mid d(v,v) = m) \\
&= (1+o(1))\alpha p_m \frac{t}{\eta} \binom{l+m+\nu/\eta-1}{l} \frac{\Gamma(m+(\nu+1)/\eta)\ l!}{\Gamma(l+1+m+(\nu+1)/\eta)}.
\end{aligned}$$

This follows by summing (13) over v for fixed l, which approximates $(t/\eta)B(\xi+1/\eta, l+1)$.

Define $n(l,m) = \mathbf{E}\,(D_t(l,m)/|V(G(t))|)$. Then $n(l,m)$ is the expected proportion of vertices of degree l at step t, with initial degree m. Thus $n(l,m)$ is **whp** asymptotic to

$$\widehat{n}(l,m) = \frac{p_m}{\eta} \frac{\Gamma(l+m+\nu/\eta)}{\Gamma(m+\nu/\eta)} \frac{\Gamma(m+(\nu+1)/\eta)}{\Gamma(l+1+m+(\nu+1)/\eta)}.$$

In particular from (22) we deduce that, as $l \to \infty$,

$$n(l,m) \sim \frac{p_m}{\eta} \frac{\Gamma(m+(\nu+1)/\eta)}{\Gamma(m+\nu/\eta)}\ l^{-\left(1+\frac{1}{\eta}\right)}.$$

□

Provided l is not too large, the concentration of the degree sequence follows directly from Theorem 2(b) and the Chebychev inequality. This is the proof of Theorem 1(d).

Corollary 6. *Let $\epsilon \geq 1/\log t$. Let $l = o(t^{1/4})$. Let $D_t(l,m) = |\{v : d(v,t) = l, d(v,v) = m\}|$, and let $\mu_{l,m} = \mathbf{E}\ D_t(l,m)$ then*

$$\mathbf{Pr}(|D_t(l,m) - \mu_{l,m}| \geq \epsilon\mu_{l,m}) \leq O\left(l^{5/2+1/2\eta}\frac{\log^3 t}{\sqrt{t}}\right).$$

Proof Let $\mu = \mathbf{E}\ D_t(l,m)$. Thus $\mu = C_l t l^{-(1+1/\eta)}$ if $l \to \infty$ and this is also true for l constant, and suitably chosen C_l.

Let $v_L = \mu/\log t$. Thus there are at most $\epsilon\mu$ vertices $v \leq v_L$ of degree l.

Let $v \in [v_L, t]$. Let $X_v = 1$ if $d(v,t) = l, d(v,v) = m$. Let $\mu' = \sum X_v$, then $(1-\epsilon)\mu \leq \mu' \leq \mu$. Using Theorem 2(b) and the Chebychev inequality, we see that

$$\mathbf{E} \sum {X_v}^2 \leq (\mu')^2(1 + l\delta^*) + \mu',$$

and thus

$$\mathbf{Pr}(|D_t(l,m) - \mu_{l,m}| \geq \epsilon\mu_{l,m}) \leq \frac{l\delta^*}{\epsilon^2} + \frac{1}{\epsilon^2\mu'}.$$

Now

$$l\delta^* = l^2 O\left(\sqrt{\frac{\log t}{v_L}}\right) = l^{2+\frac{1}{2}(1+1/\eta)} O\left(\frac{\log t}{\sqrt{t}}\right).$$

So assuming $\mu' \sim t l^{-(1+1/\eta)} \gg l\delta^*$ which is true for $l = o(t^{1/4})$ the result follows. □

There is nothing in the proof of Theorem 2 which is really specific to the vertex v. For $u \leq v$ let $d(u,v) = m_u$. Provided m_u is not too large, then $\delta^* = o(1)$ in (13). Thus, instead of estimating $\mathbf{Pr}(d(v,t) = m + l \mid d(v,v) = m)$ we can estimate $\mathbf{Pr}(d(u,t) = m_u + l \mid d(u,v) = m_u)$. This will give the proof of Theorem 1(e).

Corollary 7. *Let $\Delta(G(t))$ denote the maximum degree of $G(t)$ then*

$$\mathbf{E}\ \Delta(G(t)) = \Theta(1)t^\eta.$$

Proof

From the definition of v_0 given in (8) we can define a decreasing sequence of times $v_i, i = 0, 1, ...$ given by $(v_0,\ v_1, ..., v_i, ...)$ where $v_{i+1} = v_0(v_i)$. After $s = \Theta(\log t)$ steps we obtain $v_s = 1$.

Let $\eta < \frac{1}{2}$, and thus $\delta^* = o(1)$ in (21). We choose $v = v_0$ and $u = 1$ (say). Thus

$$\mathbf{E}\ d(1,t) \sim d(1,v_0)\left(\frac{t}{v_0}\right)^\eta.$$

Iterating back over the sequence v_i, gives

$$\mathbf{E}\ d(1,t) = \Theta(1)t^\eta.$$

The $\Theta(1)$ arises from the approximation error of (11) in the iterates of (13).

Let $\frac{1}{2} \leq \eta < 1$. In this case, $\delta^* \neq o(1)$. Assuming inductively that $d(1, v_0) = \Theta(1)(v_0)^\eta$ we divide the configuration points of vertex 1 at step v_0 into $\lceil (v_0)^{\eta/2} \rceil$ pseudo-vertices of degree $O((v_0)^{\eta/2})$ and use the previous analysis. $\square$

5 Hub-Authority process

5.1 Statement of results

The results exhibit a marked similarity to those of the Undirected Process. The basic reason for this is as follows: In the Undirected Process the transitions in degree of vertex v are described by a simple Markov process. A similar pair of processes describes transitions in in-degree and out-degree of v, and moreover these processes are effectively un-correlated.

The parameters of the model are defined similarly to the undirected case:

$$\eta^- = \frac{\alpha \overline{m}^- A_1 + \beta \overline{M} C_1}{\alpha \overline{m}^- + \gamma \overline{m}^+ + \beta \overline{M}} \qquad \nu^- = \frac{\alpha \overline{m}^- A_2 + \beta \overline{M} C_2}{\alpha + \gamma} \qquad \xi^- = m^- + \nu^-/\eta^- \tag{25}$$

$$\eta^+ = \frac{\gamma \overline{m}^+ D_1 + \beta \overline{M} B_1}{\alpha \overline{m}^- + \gamma \overline{m}^+ + \beta \overline{M}} \qquad \nu^+ = \frac{\gamma \overline{m}^+ D_2 + \beta \overline{M} B_2}{\alpha + \gamma} \qquad \xi^+ = m^+ + \nu^+/\eta^+. \tag{26}$$

Let $v_0^+, v_0^- = v_0$ of the undirected process.

Theorem 8. *(a) Degree of late vertices.*
The distribution of in- and out-degrees of any vertex are effectively uncorrelated.
Let $\{d^-(v,v) = m^-, d^+(v,v) = m^+\}$ be the initial degree of vertex v, then

$$\begin{aligned}\mathbf{Pr}(d^-(v,t) = m^- + r, d^+(v,t) = m^+ + s \mid d^-(v,v) = m^-, d^+(v,v) = m^+) = (1+o(1))\times \\ \mathbf{Pr}(d^-(v,t) = m^- + r \mid d^-(v,v) = m^-)\, \mathbf{Pr}(d^+(v,t) = m^+ + s \mid d^+(v,v) = m^+).\end{aligned}$$

(b) Expected value, distribution of in- and out-degree and proportional degree sequence of late vertices are as in the undirected case. For example:

(i)

$$\begin{aligned}\mathbf{Pr}(d^-(v,t) =& m^- + l \mid d^-(v,v) = m^-) \\ =& \binom{l + \xi^- - 1}{l} \left(\frac{v}{t}\right)^{\eta^- m^- + \nu^-} \left(1 - \left(\frac{v}{t}\right)^{\eta^-} (1+o(1))\right)^l + O\left(\frac{1}{t^K}\right).\end{aligned}$$

(ii) Let $N^-{}_t(l,m)$ denote the proportion of vertices of $G(t)$ of in-degree $l+m$ with initial in-degree m. Let

$$n^-(l,m) = \frac{\gamma\, 1(m=0) + \alpha\, 1(m>0) p_m^-}{\alpha+\gamma} \frac{1}{\eta^-} \frac{\Gamma(m + (\nu^-+1)/\eta^-)}{\Gamma(m+\nu^-/\eta^-)} \frac{\Gamma(l+m+\nu^-/\eta^-)}{\Gamma(m+l+1+(\nu^-+1)/\eta^-)}. \tag{27}$$

Then **whp** $N^-{}_t(l,m) \sim n^-(l,m)$. *In particular, as* $l \to \infty$,

$$n^-(l,m) \sim \tfrac{\gamma\, 1(m=0)+\alpha\, 1(m>0)p_m^-}{\alpha+\gamma}\,\tfrac{1}{\eta^-}\,\tfrac{\Gamma(m+(\nu^-+1)/\eta^-)}{\Gamma(m+\nu^-/\eta^-)}\; l^{-\left(1+\frac{1}{\eta^-}\right)}.$$

(iii) Let $b = \eta^+/\eta^-$. *Let* $C_{r,s} = \Theta(1)$, *constant. Let* $n_t(r,s \mid m^-, m^+)$ *be the proportion of vertices at step* t *with in-degree* r, *out-degree* s, *and initial degree* (m^-, m^+). *Then, provided* $r,s = O(?)$ **whp** $n_t(r,s \mid m^-, m^+)$ *is given as follows:*
If $r \to \infty$ *and* $s = O(r^b)$ *then*

$$n_t(r,s \mid m^-, m^+) = C_{r,s} r^{-\left(2+\frac{\eta^+}{\eta^-}\xi^+ + \frac{1}{\eta^-}\right)} s^{-\left(2-\xi^+\right)}. \tag{28}$$

If $s \to \infty$ *and* $r = O(s^{1/b})$ *then*

$$n_t(r,s \mid m^-, m^+) = C_{r,s} r^{-\left(2-\xi^-\right)} s^{-\left(2+\frac{\eta^-}{\eta^+}\xi^- + \frac{1}{\eta^+}\right)}. \tag{29}$$

If $s = \Theta(r^b)$ *then the expressions (28), (29) are equivalent.*

5.2 Proof of Theorem 8

The analysis is almost identical with the undirected case. We only give an outline description of the proof, except in 8(iii), where details differ.

Noting that an α-event at step t (NEW-IN procedure) results in the increase of the in-degree of $G(t-1)$ etc, the variables $X^-(t+1)$ (resp. $X^+(t+1)$) for the alteration in $d^-(v,t)$ (resp. $d^+(v,t)$) at step $t+1$ are distributed as follows:

$$\begin{aligned} X^-(t+1) &\sim 1_\alpha(t)\mathrm{Bin}(m^-(t), P_A(t)) + 1_\beta(t)\mathrm{Bin}(M(t), p_C(t)) \\ X^+(t+1) &\sim 1_\gamma(t)\mathrm{Bin}(m^+(t), P_D(t)) + 1_\beta(t)\mathrm{Bin}(M(t), p_B(t)). \end{aligned}$$

Let $d^-(v,v) = 1_\alpha(v)m^-$, $d^-(v,t) = 1_\alpha(v)m^- + l^-$ etc. The values of $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_{l^-})$, $\boldsymbol{\tau} = (\tau_1, ..., \tau_{l^+})$ for the transition times of the in- and out-degrees of vertex v are unrelated except that the procedures NEW-IN and NEW-OUT are mutually exclusive. Thus any transition times in $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ occurring via these procedures are distinct. As we only analyse cases where the number of transitions l^-, l^+ are small compared to the number of steps t the distributions of $d^-(v,t)$ and $d^+(v,t)$ are effectively uncorrelated. Detailed reworking shows that the error is of the same order of magnitude as the errors in the proof of Theorem 2.

The specific transition probabilities at step $t+1$ are calculated as follows:
For $\mathbf{Pr}(X^-(t+1) = 0,\ X^+(t+1) = 0)$ the calculations corresponding to those made in (16)-(18) are

$$\begin{aligned} &\mathbf{Pr}(X^-(t+1) = 0,\ X^+(t+1) = 0 \mid d^-(v,t) = m^- + l^-,\ d^+(v,t) = m^+ + l^+) \\ &\quad = \mathbf{E}\ \left(1_\alpha(1-p_A)^{m^-(t+1)} + 1_\beta[(1-p_B)(1-p_C)]^{M(t+1)} + 1_\gamma(1-p_D)^{m^+(t+1)}\right) \\ &\quad = 1 - \left(\frac{\eta^-(m^- + j^-) + \nu^-}{t} + \frac{\eta^+(m^+ + j^+) + \nu^+}{t}\right)\left(1+\epsilon(j^-, j^+, t)\right), \end{aligned}$$

where

$$\epsilon(j^-,j^+,t) = O\left(\sqrt{\log t/t} + [(m^- + j^-) + (m^+ + j^+)]/t\right).$$

The calculations for $\mathbf{Pr}(X^-(t+1) = 1, X^+(t+1) = 0)$, $\mathbf{Pr}(X^-(t+1) = 0, X^+(t+1) = 1)$ are likewise similar to those of Theorem 2. Higher order transitions of the form $(X^- + X^+ \geq 2)$ are second order terms with probability given by

$$\mathbf{Pr}(X^- + X^+ \geq 2) = \mathbf{Pr}(X^- + X^+ = 1) O\left(\frac{j^- + j^+}{t}\right).$$

It follows that, provided we substitute the values $\eta^-, \eta^+, \nu^-, \nu^+$ given above for η, ν of Section 3, those results will follow unchanged.

Proof of 8(iii). Let $N_t(r,s) = N_{G(t)}(r, s \mid m^-, m^+)$ denote the number of vertices of $G(t)$ with in-degree r, out-degree s, and initial in-degree m^- and out-degree m^+. Thus

$$\mathbf{E}\, N(r, s \mid m^-, m^+) = \sum_{v=1}^{t} \mathbf{Pr}(d^-(v,t) = m^- + r, d^+(v,t) = m^+ + s \mid m^-, m^+).$$

By (???) the value of $N_t(r,s)$ is strongly concentrated, and the proportion $n(r,s)$ of such vertices is given **whp** by $(1+o(1))\mathbf{E}\, N_t(r,s)/((\alpha+\gamma)t)$. By 8(a), (b)(i), we have

$$\begin{aligned}
\mathbf{E}\,(N)_t\,(r,s) &= O(v_0) + (1+o(1)) \sum_{v=v_0}^{t} \mathbf{Pr}(d^-(v,t) = m^- + r \mid m^-)\mathbf{Pr}(d^+(v,t) = m^+ + s \mid m^+) \\
&= C r^{-(2+\xi^-)} s^{-(2+\xi^+)} I(r,s).
\end{aligned}$$

The value of $I(r,s)$ is given by

$$I(r,s) = \int_0^1 x^a (1-x)^r (1-x^b)^s dx,$$

where $b = \eta^+/\eta^-$ and $a = \eta^+/\eta^-\ \xi^+ + 1/\eta^- - \xi^-$. Let $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ denote the standard Beta integral. Thus $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$.

Assuming at least one of $r, s \to \infty$ there are three cases to consider: $s = o(r^b)$, $s = \Theta(r^b)$ and $r = o(s^{1/b}$.

Case $s = o(r^b)$. The main measure of $\int_0^1 x^a(1-x)^r dx$ comes from the interval $[0, \omega/r]$ where $\omega \to \infty$ arbitrarily slowly. Let $s = (r/\omega^2)^b$, thus when $x = \omega/r$, $sx^b = 1/\omega^b$ and so $(1-x^b)^s = (1-o(1))$. Thus

$$I(r,s) = (1-o(1))B(a+1, r+1).$$

Case $r = o(s^{1/b})$. This is similar to the above case. Making the transformation $y = x^b$ we find

$$I(r,s) = \frac{1}{b}(1-o(1))B((a+1)/b, s+1).$$

Case $s = \Theta(r^b)$**.** Clearly $(1 - x^b) \leq 1$ so

$$I(r,s) \leq B(a+1, r+1) = Cr^{-(a+1)}.$$

To obtain a lower bound we argue as follows: As $s = \Theta(r^b)$, we can write $s = dr^b$ where d is constant. Let $x_0 = c/r$, where $d = c^{1-b}$, then x_0 satisfies $rx = sx^b$. Assuming that $sx^b \leq rx$ for $x \leq x_0$, and using $e^{-y/(1-y)} \leq 1 - y$, we have

$$\begin{aligned} I(r,s) &\geq \int_0^{x_0} x^a e^{-3rx} dx + \int_{x_0}^{\omega} x^a e^{-3sx^b} dx \\ &= \frac{1}{(3r)^{a+1}} \int_0^c y^a e^{-y} dy + \frac{1}{b(3s)^{(a+1)/b}} \int_c^{\omega'} y^{\frac{a+1}{b}-1} e^{-y} dy. \end{aligned}$$

Thus

$$I(r,s) \geq C_1 r^{-(a+1)} + C_2 s^{-(a+1)/b}.$$

However, as $r = s^{1/b} c^{1-1/b}$ this is equivalent to $I(r,s) \geq Cr^{-(a+1)}$. Combining this with the upper bound, the result follows.

5.3 Generalized Hub-Authority Process

The Hub-Authority Process was formulated in terms of three exclusive events, namely the NEW-IN, NEW-OUT and OLD procedures. It is perfectly straightforward to coalesce the NEW-IN and NEW-OUT procedures back to a single process which occurs with probability $\alpha^* = \alpha + \gamma$. We endow this process with a finite distribution $\boldsymbol{p} = (p_{i,j},\ i \geq 0, j \geq 0,\ i + j \geq 1)$. Define

$$\overline{m}^- = \sum_{i,j} i p_{i,j} \qquad \overline{m}^+ = \sum_{i,j} j p_{i,j}.$$

Our original Hub-Authority Process has $\boldsymbol{p} = (p_{i,0}, i \geq 1,\ p_{0,j}, j \geq 1)$ as joint distribution, and marginals $\sum_i p_{i,0} = \alpha/(\alpha + \gamma)$ etc. The change in parameters (25) for the general case is (eg)

$$\eta^- = \frac{\alpha^* \overline{m}^- A_1 + \beta \overline{M} C_1}{\alpha^*(\overline{m}^- + \overline{m}^+) + \beta \overline{M}} \qquad \nu^- = \frac{\alpha^* \overline{m}^- A_2 + \beta \overline{M} C_2}{\alpha^*}.$$

6 Undirected hypergraph processes

Let $\boldsymbol{\pi} = (\pi_2, ..., \pi_r)$ be the distribution of edge sizes for the NEW procedure, and $\boldsymbol{\rho} = (\rho_2, ..., \rho_R)$ the distribution of edge sizes for the OLD procedure. Let $\boldsymbol{\pi}, \boldsymbol{\rho}$ be finite, let $\overline{\pi} = \sum j\pi_j$ and let $\overline{\rho} = \sum j\rho_j$. The distribution of $X = X(t+1)$ is given by

$$X \sim 1_\alpha(t) \left(\textstyle\sum_{i=1}^{m(t)} \mathrm{Bin}(R_{N,i} - 1, p_A(t)) \right) + 1_\beta(t) \left(\mathrm{Bin}(M(t), p_B(t)) + \textstyle\sum_{i=1}^{M(t)} \mathrm{Bin}(R_{O,i} - 1, p_C(t)) \right),$$

where $R_{N,i}$ (resp. $R_{O,i}$) is a random variable giving the size of an edge in the NEW (resp OLD) procedure.

By taking $\mathbf{E}\, X(t)$ within the set $I(t)$ of good histories, we see that the parameters η, ν of the process are as follows:

$$\eta = \frac{\alpha\overline{m}(\overline{\pi}-1)A_1 + \beta\overline{M}(B_1 + (\overline{\rho}-1)C_1)}{\alpha\overline{m}\overline{\pi} + \beta\overline{M}\overline{\rho}} \qquad \nu = \frac{\alpha\overline{m}(\overline{\pi}-1)A_2 + \beta\overline{M}(B_2 + (\overline{\rho}-1)C_2)}{\alpha}.$$

The details of the proof and the results are identical with those of Theorem 1.

7 Equivalence between models

The general definition of the web-graph process is unavoidably somewhat complicated. However, the asymptotic degree sequence of any undirected web-graph or scale-free graph obeys a power law. That is, the proportion of vertices of degree k tends to Ck^{-x} as $k \to \infty$, where x is the power law parameter. In the notation of this paper, $x = 1 + 1/\eta$, $(0 < \eta < 1)$. The secondary details are hidden in the model specific constant, C. We observe that for undirected web-graphs, it is impossible to generate processes with parameter $\eta > \frac{1}{2}$, (ie: $x > 3$), based solely on the addition of NEW vertices at each step.

Any undirected web-graph process with parameter η is equivalent to a simple *amoritzed* process with the same parameter. This amortized process inserts a single edge at each step.

(a) For $\eta \leq \frac{1}{2}$ the amortized process is a NEW tree-process in which a single edge is added at each step. The copy probability (A_1) of the new edge is set to $A_1 = 2\eta$, and the uar probability to $1 - 2\eta$.

(b) For $\frac{1}{2} \leq \eta < 1$ the amortized process is a mixed OLD, NEW copy-only process in which a single edge is added at each step. The probability α of a NEW vertex is $\alpha = 2(1-\eta)$.

We pause to remark that amortization of the process hides a multitude of very different models and structures. For example, models in which the number of edges generated at each step has a fixed value m or average values $\overline{m} = \overline{M}$ are all equivalent to the same one edge model.

The Linear Chord Diagram (LCD) process described in [2] is an elegant configuration model, and has $\eta = \frac{1}{2}$. Hence it is equivalent to a random tree in which the copy rule is always followed. However the LCD process with one edge consists of a directed forest of $\Theta(\log t)$ components, where each component is rooted at a looped vertex.

The papers [12], [13] propose a web-graph model in which the NEW procedure with one edge is followed at each step. Thus they describe the amortized process with $\eta \leq \frac{1}{2}$.

The models of [9] and [10] are the generalization of scale-free (LCD) model to incorporate uar choices, and are equivalent to an amortized model of type $\eta = 1/(2(a+1))$, where a is the *initial attractiveness*. In other words, the copy choice is made by an edge with probability $1/(a+1)$ and the uar choice with probability $a/(a+1)$.

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