Homework 7: due December 4

1. Show that if X denotes the number of cycles in a random permutation of $[n]$ then $\mathbb{P}(X \geq t) \leq \mathbb{P}(Binomial(t, 1/2) \leq \lceil \log_2 n \rceil)$. Deduce that for every constant $L > 0$, there exists a constant $K > 0$, such that $\mathbb{P}(X \geq K \log n) \leq n^{-L}$.
(Hint: consider the size of the cycle C containing vertex 1 and the number of cycles in the remaining permutation on $[n] \setminus C$.)

2. For a graph $G = (V, E)$ let $f : V \to V$ be a G-mapping if $(v, f(v))$ is an edge of G for all $v \in V$. Let G be a graph with n vertices and minimum degree $(\frac{1}{2} + \varepsilon)n$ for some fixed $\varepsilon > 0$. Let $H = \bigcup_{i=1}^{k} H_i$ where (i) $k = O(1)$ is sufficiently large, and (ii) $H_1, H_2, \ldots, H_k$ are independent uniform random G-mappings. Show that w.h.p. H is connected.

3. In the random digraph $G_{k-in,k-out}$ each $v \in [n]$ independently chooses k uniformly random in-neighbors and k uniformly random out-neighbors. Show that $G_{k-in,k-out}$ is k-strongly connected for $k = O(1)$ is sufficiently large.