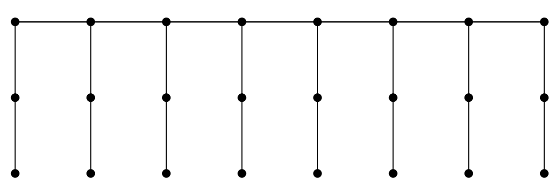
Homework 4: due October 7

In all of these questions, the graph in question is $G_{n,p}$.

1. Show that if three divides n and $np^2 \gg \log n$ then w.h.p. $G_{n,p}$ contains $n/3$ vertex disjoint triangles. (Hint: Randomly partition $[n]$ into three sets A, B, C of size $n/3$. Choose a perfect matching M between A and B and then match C into M).

   Solution: Because $p \gg \frac{\log n}{n}$, there will w.h.p. be a perfect matching M between A and B. Now define a new random bipartite graph with vertices M and C and an edge beteen $e = \{x, y\} \in M$ and $v \in C$ iff v is adejacent to x and y. This is distributed as the bipartite graph $G_{n/3,n/3,p^2}$ and this will have a perfect matching w.h.p., giving us $n/3$ disjoint trangles.

2. Show that if $p = \frac{(6+\varepsilon)\log n}{n}$ for $\varepsilon > 0$ constant, then w.h.p. $G_{n,p}$ contains a copy of the following caterpillar on n vertices. The diagram below is the case $n = 24$.



   Solution: Write $G_{n,p}$ as $G_{n,p_1} \cup G_{n,p_2} \cup G_{n,p_3}$ where $p_2 = p_3 = \frac{(1+\varepsilon/10)\log n}{n/3}$. $p_1 = \Omega(\log n/n)$ and so w.h.p. G_{n,p_1} will have a path P of length $n/3$. Divide $[n] \setminus V(P)$ into A, B. Then w.h.p. there is a perfect matching in G_{n,p_2} between A and $V(P)$ and also a perfect matching in G_{n,p_3} between A and B

3. Suppose that H is obtained from $G_{n,1/2}$ by planting a clique C of size $m = n^{1/2}\log n$ inside it. describe a polynomial time algorithm that w.h.p. finds C. (Think that an adversary adds the clique without telling you where it is).

   Solution: W.h.p., the vertices in C will have degree at least $n/2 + m - O((n\log n)^{1/2})$ and the vertices not in C will have degree at most $n/2 + O((n\log n)^{1/2})$ and so we can find C by checking the degree sequence. In fact we only need to take the $10\log n$ vertices of highest degree A and then C will be the set of vertices that are adjacent to all of A.