Homework 3: due September 25

In all of these questions, the graph in question is $G_{n,p}$.

1. Let $k \geq 3$ be fixed and let $p = \frac{c}{n}$. Show that there exists $\theta = \theta(c,k) > 0$ such that w.h.p. all vertex sets S with $|S| \leq \theta n$ contain fewer than $k|S|/2$ edges. Deduce that w.h.p. either the k-core of $G_{n,p}$ is empty or it has size at least θn.

   Solution: We compute the expected number of sets of size $s \leq \theta n$ that contain at least $ks/2$ edges. But if the k-core has s vertices then it has at least $ks/2$ edges.

   $$\sum_{s=k}^{\theta n} \binom{n}{s} \binom{\binom{s}{2}}{ks/2} \left(\frac{c}{n}\right)^{ks/2} \leq \sum_{s=k}^{\theta n} \left(\frac{ne}{s}\right)^s \left(\frac{es^2}{ks}\right)^{ks/2} \left(\frac{c}{n}\right)^{ks/2} = \sum_{s=k}^{n} \left(\left(\frac{s}{n}\right)^{1-k/2} \cdot \frac{e^{k/2+1}c^{k/2}}{k^{k/2}}\right)^s = o(1),$$

   if $\theta^{1-k/2} \leq \frac{k^{k/2}}{2e^{k/2+1}c^{k/2}}$. So if $s \leq \theta n$ then w.h.p. there are no sets containing $ks/2$ edges.

2. Let m_1^* be the hitting time for minimum degree 1 in the graph process. Suppose that $e_{m_1^*} = \{u, v\}$ where v is the vertex whose only incident edge is $e_{m_1^*}$. Show that w.h.p. there is no triangle containing u.

   Solution: The vertex u is equally likely to be any vertex distinct from v. We know that w.h.p. G_{n,M_1^*} is contained in $G_1 = G_{n,n\log n}$. The expected number of triangles in G_1 is $O(\log^3 n)$ and so the Markov inequality implies that there are $O(\log^4 n)$ w.h.p. Thus the probability that u is in a triangle is $O(\log^4 n/n) = o(1)$.

3. Let $G_{n,n,p}$ be the random bipartite graph with vertex bi-partition $V = (A, B)$, $A = [1, n]$, $B = [n+1, 2n]$ in which each of the n^2 possible edges appears independently with probability p. Let $p = \frac{\log n + \omega}{n}$, where $\omega \to \infty$. Show that w.h.p. $G_{n,n,p}$ is connected.

   Solution:

   $$\begin{aligned} \mathbb{P}(G_{n,n,p} \text{ is not connected}) &\leq 2\sum_{k=1}^{n/2}\sum_{\ell=1}^{n} \binom{n}{k}\binom{n}{\ell} k^{\ell-1}\ell^{k-1}p^{k+\ell-1}(1-p)^{k(n-\ell)+\ell(n-k)} \\ &\leq 2\sum_{k=1}^{n/2}\sum_{\ell=1}^{n} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{\ell}\right)^\ell k^{\ell-1}\ell^{k-1}p^{k+\ell-1}(1-p)^{k(n-\ell)+\ell(n-k)} \\ &\leq \frac{1}{p}\sum_{k=1}^{n/2}\sum_{\ell=1}^{n} (e^{1+\ell p/2}p)^k (e^{1+kp/2}p)^\ell e^{-(k+\ell)\omega} = o(1). \end{aligned}$$