Homework 2: due September 11

1. Suppose that $0 < p < 1$ is constant. Show that w.h.p. $G_{n,p}$ has diameter two.

   Solution: First of all note that w.h.p. there exist pairs v, w that are non-adjacent. ($\mathbb{P}(G_{n,p} = K_n) = (1-p)^{\binom{n}{2}} = o(1)$.) So w.h.p. the diameter is at least 2.

   Now if for every pair of non-adjacent vertices v, w we can find a vertex x such that $\{v, x\}$ and $\{w, x\}$ are edges then the diameter is at most 2. So,

$$\mathbb{P}(\text{Diameter}(G_{n,p}) > 2) \leq \binom{n}{2}(1-p^2)^{n-2} = o(1).$$

2. Let $f : [n] \to [n]$ be chosen uniformly at random from all n^n functions from $[n] \to [n]$. Let $X = \{j : \nexists i \ s.t. \ f(i) = j\}$. Show that w.h.p. $|X| \approx e^{-1}n$.

   Solution:

$$\mathbb{E}(|X|) = n\left(1 - \frac{1}{n}\right)^n = e^{-1}n + O(1).$$

$$\mathbb{E}(|X|(|X|-1)) = \sum_{i=1}^{n} \sum_{j=1, j\neq i} \left(1 - \frac{2}{n}\right)^n$$

$$= n(n-1)\left(1 - \frac{2}{n}\right)^n \leq \mathbb{E}(|X|)^2.$$

   Then

$$\mathbb{P}(|\,|X| - \mathbb{E}(|X|\,| \geq \varepsilon|X|) \leq \frac{\mathbb{E}(|X|^2) - \mathbb{E}(|X|)^2}{\varepsilon^2 \mathbb{E}(|X|)^2} \leq \frac{1}{\varepsilon^2 \mathbb{E}(|X|)}.$$

   Putting $\varepsilon = n^{-1/3}$ we see that w.h.p. $|X| = e^{-1}n + O(n^{2/3})$.

3. Suppose that $p = \frac{c}{n}$ where $c > 1$ is a constant. Show that w.h.p. the giant component of $G_{n,p}$ is non-planar. (Hint: Argue that the number of edges in the giant is asymptotically equal to $(c+x)/2$ times the number of vertices in the giant. You can assume that $c + x > 2$. Remove edges from the giant so that the girth is large. Now use Euler's formula for the case when the graph has large girth. Note also that an ν-vertex planar graph of girth g has at most $\nu(1 - 2/g)^{-1}$ edges.)

   Solution: The density of the giant (ratio of edges to vertices) is w.h.p. asymptotically equal to $((1 - x^2/c)cn/2)/(1 - x/c) = (c + x)/2 = 1 + \xi$ where $\xi = \xi(c) > 0$. The expected number of edges X on cycles of length at most g is

$$\sum_{i=3}^{g} \binom{n}{i} \frac{(i-1)!}{2} \left(\frac{c}{n}\right)^i \sim \sum_{i=3}^{g} \frac{c^i}{2i} < c^g.$$

   The Markov inequality implies that $X = O(\log n)$ w.h.p. Let $g = 3/\xi$. Then after removing $O(\log n)$ edges we have a graph of girth at least g and with density $\sim (1 + \xi) > (1 - 2/g)^{-1}$. This is not planar.