Homework 7: Solutions

7.6.9 Suppose that H is obtained from $G_{n,1/2}$ by planting a clique C of size $m = n^{1/2}\log n$ inside it. describe a polynomial time algorithm that w.h.p. finds C. (Think that an adversary adds the clique without telling you where it is).

Solution: First observe that w.h.p. all vertices of $G_{n,1/2}$ have degrees in the range $n/2 \pm O((n\log n)^{1/2})$. This remains true in H for vertices not in C. Vertices in C will w.h.p. have their degrees increased by $(1/2 - o(1))n^{1/2}\log n$ and so they will constitute the $n^{1/2}\log n$ vertices of largest degree in H.

7.6.10 Show that if $d > 2k\log k$ for a positive integer $k \geq 2$ then w.h.p. $G(n, d/n)$ is not k-colorable.

Hint:Consider the expected number of proper k-coloring's:

$$\sum_{n_1+\cdots+n_k=n} \left(\frac{n!}{n_1!\cdots n_k!}\right)(1-p)^{\sum_i n_i(n_i-1)/2}$$

Solution: Let $d = 2k\log k + 2\epsilon$. The expected number of k-colorings can be bounded by

$$\begin{aligned}
&\sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k}(1-p)^{\sum_{i=1}^k \binom{n_i}{2}} \\
&\leq \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \exp\left\{-\frac{d}{n}\sum_{i=1}^k \binom{n_i}{2}\right\} \\
&\leq \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \exp\left\{-\frac{d}{n}\cdot k\binom{n/k}{2}\right\} \\
&\leq \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \exp\left\{-(\log k + \epsilon)n\right\} \\
&\leq k^n \exp\left\{-(\log k + \epsilon)n\right\} \\
&= o(1).
\end{aligned}$$

9.4.2 Show that for every $\epsilon > 0$ there exists $c_\epsilon > 0$ such that the following is true w.h.p. If $c \geq c_\epsilon$ and $p = c/n$ and we remove any set of at most $(1-\epsilon)cn/2$ edges from $G_{n,p}$, then the remaining graph contains a component of size at least $\epsilon n/4$.

Solution: Let $m = 1/\epsilon$ and let $S_1, S_2, \ldots, S_m$ be a partition of $[n]$ into sets of size between $\epsilon n/4$ and $\epsilon n/2$. (We will omit rounding errors to simplify the explanation.) In K_n, there are at least $\nu_\epsilon = \binom{n}{2} - \frac{\epsilon n^2}{4} \sim \frac{n^2}{2}\left(1 - \frac{\epsilon}{2}\right)$ edges that join different sets in the partition. Call these edges *joiners.*

Now there are at most $\left(\frac{4}{\epsilon}\right)^n$ such partitions and so using the Chernoff bounds we see that for $\eta = \epsilon/4$

$$\mathbf{P}(\exists\text{partition with fewer than } (1-\eta)c\nu_\epsilon/n \text{ joiners}) \\ \leq \left(\frac{4}{\epsilon}\right)^n \exp\left\{-\frac{c\eta^2}{4}\left(1-\frac{\epsilon}{2}\right)\right\} = o(1),$$

for large enough c.

So w.h.p., after deletion of edges, every such partition still contains at least

$$\frac{(1-\eta)c\nu_\epsilon}{n} - \frac{(1-\epsilon)cn}{2} = \frac{cn}{2}\left(\left(1-\frac{\epsilon}{4}\right)\left(1-\frac{\epsilon}{2}\right) - (1-\epsilon)\right) > 0.$$

But if there is no component of size at least $\epsilon n/4$ after deletion of edges, then we can find such a partition with no joiners.