Homework 6: Solutions

6.7.12 Write $G_{n,p} = G_{n,p_1} \cup G_{n,p_2}$ where $p_1 = \frac{100}{n}$ and $(1-p) = (1-p_1)(1-p_2)$ and $p_2 \sim p$.

Now Theorem 6.8 implies that w.h.p. G_{n,p_1} contain a path of length greter than $n/2$. And then Theorem 6.1 implies that w.h.p. we can complete this path to a caterpillar.

6.7.16 Let T' be obtained from T by removing the leaves. Then write $G_{n,p} = G_{n,p_1} \cup G_{n,p_2}$ where $p_1 = p_2 \sim \frac{K \log n}{2n}$ and $1 - p = (1 - p_1)(1 - p_2)$. Build a copy of T' in G_{n,p_1} as follows: fix one vertex v of T' as a root and then do a breadth first search to construct $V_I, i = 1, 2, \ldots$ where V_i is the set of vertices at distance i from v. We then embed T' into G_{n,p_1} in the order $V_0, V_1, V_2, \ldots$. Suppose that we have embedded $V_0, V_1, \ldots, V_i$ as $W_0, W_1, \ldots, W_i$. Then to create V_{i+1} we must for each $v \in V_i$ find up to $c_1 \log n$ neighbors from a set of size at least $c_2 n$. This will always be possible with probability $1 - o(n^{-1})$ and so we succeed in embedding T'.

After this, we can use Theorem 6.1 to find a matching that will allow us to add the leaves to create T.

6.7.17 Running DFS on the graph G_R induced by the red edges, we see that if there is no red path of length $n/1000$ then we find sets D, U, A with $|D| = |U| \geq \frac{999n}{2000}$ such that there is no red edge between D and U. Similarly, $[n]$ can be partitioned into D', U', A' such that $|D'| = |U'| \geq \frac{999n}{2000}$ and there is no blue edge between D' and U'.

Let $X = U \cap U', Y = U \cap D', X' = D \cap U', Y' = D \cap D'$ and let $x = |X|, y = |Y|, x' = |X'|, y' = |Y'|$. Then

$$x + y = |U \cap (U' \cup D')| = |U \setminus A'| \geq \frac{999n}{2000} - \frac{n}{1000} = \frac{997n}{2000}. \tag{1}$$

Similarly,

$$x' + y', x + x', y + y' \geq \frac{997n}{2000}. \tag{2}$$

It follows that either (i) $x, y' \geq \frac{997n}{4000}$ or (ii) $x', y \geq \frac{997n}{4000}$. (Failure of (i) and (ii) implies that (1) or (2) fail.) Suppose then that $x', y \geq \frac{997n}{2000}$. Now $X' \subseteq D$ and $Y \subseteq U$ and so there are no $X' : Y$ red edges. Furthermore, $X' \subseteq U'$ and $Y \subseteq D'$ and so there are no $X' : Y$ blue edges either. In other words $X' : Y = \emptyset$. But,

$$\mathbf{P}\left(\exists \text{ disjoint } S, T : |S|, |T| \geq \frac{997n}{4000} \text{ and } S : T = \emptyset\right)$$
$$\leq 2^{2n} \left(1 - \frac{1000}{n}\right)^{(997n/4000)^2} = o(1).$$