Homework 5: Solutions

6.7.4 Applying Hall's theorem and defining $L = \lceil 2 \log n \rceil$,

$$\begin{aligned}
\mathbf{P}(\text{no p.m.}) &\leq \sum_{k=L}^{n} \binom{n}{k} \binom{n}{k-1} \left(\frac{\binom{k-1}{L}}{\binom{n}{L}} \right)^k \\
&\leq \sum_{k=L}^{n} \left(\frac{ne}{k} \right)^k \left(\frac{ne}{k-1} \right)^{k-1} \left(\frac{k-1}{n} \right)^{Lk} \\
&\leq \sum_{k=L}^{n} \left(\left(\frac{k}{n} \right)^{L-2} \cdot e^2 \cdot \left(\frac{k-1}{k} \right)^L \right)^k \\
&= \sum_{k=L}^{n} u_k.
\end{aligned}$$

Now if $k = (1-\epsilon)n$ then we have $(k/n)^{L-2} \leq e^{-\epsilon(L-2)} \leq e^{-3+o(1)}$ if $\epsilon \geq 3/L$. We deduce from this that

$$\sum_{k=L}^{n_0} u_k = o(1) \text{ where } n_0 = n - \frac{4n}{L}.$$

So we need to be more careful for $k > n_0$. If there is a set of size k on one side of the partition with at most $k-1$ neighbors, then there is a set X of size $\ell = n - k + 1$ on the other side of the partition with at most $\ell - 1$ neighbors. We estimate this by

$$\binom{n}{\ell-1} \mathbf{P}\left(Bin\left(n, 1 - \left(1 - \frac{\ell}{n}\right)^L \right) \leq \ell - 1 \right). \tag{1}$$

We choose X in $\binom{n}{\ell-1}$ ways and then $1 - \left(1 - \frac{\ell}{n}\right)^L$ lower bounds the probability that a vertex chooses a neighbor in X.

Summing over $\ell \leq \ell_0 = n - n_0 + 1$, this is at most

$$\begin{aligned}
&2 \sum_{\ell=1}^{\ell_0} \binom{n}{\ell-1}^2 \left(1 - \left(1 - \frac{\ell}{n} \right)^L \right)^{\ell-1} \left(1 - \frac{\ell}{n} \right)^{L(n-\ell+1)} \qquad (2) \\
\leq& 2 \sum_{\ell=1}^{\ell_0} \left(\frac{ne}{\ell-1} \right)^{2\ell-2} \left(\frac{\ell L}{n} \right)^{\ell-1} e^{-L\ell(1+o(1))} \\
\leq& 2 \sum_{\ell=1}^{\ell_0} \left(\frac{ne^2 \ell L}{\ell-1} \right)^{\ell-1} n^{-(2-o(1))\ell} \\
=& o(1).
\end{aligned}$$

Explanation for (2)**:** the binomial probability in (1) is dominated by the $\ell - 1$ term, leading to the factor 2.

6.7.9 Following the hint we partition $[n]$ into 3 sets A, B, C of size $n/3$. The bipiartite graph H induced by A, B is distributed as $G_{n/3,n/3,p}$ and since $\frac{n}{3}p \gg \log \frac{n}{3}$ this graph has a perfect matching w.h.p. Fix a perfect matching M of H and define another random bipartite graph K with **vertices** M, C and an edge (e, x) for each $e = \{u, v\} \in M, x \in C$ such that the edges $\{x, u\}, \{x, v\}$ both exist. The random graph K is distributed as $G_{n/3,n/3,p^2}$ and since $\frac{n}{3}p^2 \gg \log \frac{n}{3}$ this graph has a perfect matching w.h.p. This perfect matching corresponds to $n/3$ vertex disjoint triangles.

6.7.10 Arguing as in the proof of Theorem 6.1 we see that

$$\mathbf{E}Y \leq 2 \sum_{k=2}^{n/2} \binom{n}{k} \binom{n}{k-1} \binom{k(k-1)}{2k-2} p^{2k-2} (1-p)^{k(3n/4-k)}$$

The only change here is that we can only guarantee that S has at least $k(3n/4 - k)$ neighbors not in T. Continuing,

$$\begin{aligned}
\mathbf{E}Y &\leq 2 \sum_{k=2}^{n/2} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{k-1}\right)^{k-1} \left(\frac{Kke \log n}{2n}\right)^{2k-2} n^{-Kk(3/4-k/n)} \\
&\leq \frac{n^2}{\log^2 n} \sum_{k=2}^{n/2} \left(\frac{ne}{k} \cdot \frac{ne}{k-1} \cdot \left(\frac{Kke \log n}{2n}\right)^2 \cdot n^{-K/2}\right)^k \\
&= o(1),
\end{aligned}$$

if $K \geq 4$.