Homework 4: Solutions

3.3.2 Let Z_k denote the number of vertices of degree k in the giant component of $G_{n,p}$. Fix a vertex, say vertex n and consider $H_1 = G_{n-1,p}$. We know that w.h.p. H_1 will have a giant component C of size $\approx \left(1 - \frac{x}{c}\right) n$. Then, n will be part of the giant component of $G_{n,p}$ and have degree k iff (i) it has degree k and (ii) at least one its neighbors is in C. Let p_k denote the probability of (i), (ii). (This assumes that H_1 has a giant and this fails to happen with probability $o(1) = o(p_k)$.) Now the probability of (i) is asymptotically equal to $\frac{ce^{-c}}{k!}$ and given (i), the neighbors of n will be a uniform random k-subset of $[n-1]$. And so the probability (ii) fails to hold, given (i) is asymptotically equal to $\left(\frac{x}{c}\right)^k$. Thus p_k is asymptotically equal to $\frac{c^k e^{-c}}{k!}\left(1 - \left(\frac{x}{c}\right)^k\right)$. This shows that $\mathbf{E}(Z_k) \approx np_k$.

To show $Z_k \approx np_k$ w.h.p. we can use the second moment method. For this we consider $H_2 = G_{n-2,p}$ and the probability that both $n, n-1$ are degree k vertices of the giant. This is at most $p + (1 + o(1)p_k^2 = (1 + o(1))p_k^2$, where p accounts for $n, n-1$ being adjacent. Thus $\mathbf{Var}(Z_k) \approx \mathbf{E}(Z_k)^2$ and the Chebyshev inequality finishes the proof.

4.3.4 Fix $v \in [n]$. Let $\mathcal{A}_i, i = 0, \ldots, i_0 = \left\lfloor \frac{2 \log n}{3 \log d} \right\rfloor$ be the event that $|S_i(v)| \in [(d/2)^i, (2d)^i]$ for all $v \in [n]$. Clearly $\mathcal{A}_0$ is true. Now if $p = m/N$ so that $d \approx np$ then in $G_{n,p}$,

$$\mathbf{P}(\neg \mathcal{A}_{i+1} \mid S_i(v), \mathcal{A}_j, j \leq i) = \mathbf{P}\left(Bin\left((n - \sum_{j=0} i|S_j(v)|, 1 - (1-p)^{|S_i|}\right)\right).$$

Observe that $d^{i_0} \approx n^{2/3}$. Given the conditioning, $\left|\sum_{j=0} i|S_j(v)|\right| = o(n)$ and $1 - (1-p)^{|S_i|} \approx |S_i|p$. Thus,

$$\mathbf{P}(\neg \mathcal{A}_{i+1} \mid S_i(v), \mathcal{A}_j, j \leq i) = \mathbf{P}\left(Bin\left((n - o(n), (1 - o(1))|S_i|p\right)\right).$$

The expected value of the binomial in the above is $\approx d|S_i|$ and applying the Chernoff bounds we get that

$$\mathbf{P}(\neg \mathcal{A}_{i+1} \mid S_i(v), \mathcal{A}_j, j \leq i) \leq e^{-\Omega(d|S_i|)} \text{ and } d|S_i|) \gg \log n.$$

This implies that

$$\mathbf{P}(\neg \mathcal{A}_{i+1} \mid \mathcal{A}_j, j \leq i) \leq n^{-\omega} \text{ where } \omega \to \infty.$$

But, then

$$\mathbf{P}(\neg \mathcal{A}_i) \leq \mathbf{P}(\neg \mathcal{A}_1) + \mathbf{P}(\neg \mathcal{A}_2 \mid \neg \mathcal{A}_1) + \mathbf{P}(\neg \mathcal{A}_3 \mid \mathcal{A}_1, \mathcal{A}_2) \cdots \leq in^{-\omega}.$$

and we can use the union bound to deal with all choices of v, i.

4.3.5 Given $\mathcal{A}_1$ we choose $d/2$ vertices $v_1, v_2, \ldots, v_{d/2}$ in $S_1(v)$. We then argue that w.h.p. we can find at least $(d/2)^{i_0-1} = \nu_d = \Omega(n^{2/3})$ vertices V_1 at distance $i_0 - 1$ from v_1. We then repeat the argument with respect to v_2, but this time we do avoid the $O(n^{2/3})$ vertices W_1 used in the construction of V_1. The probability bounds are hardly affected by this restriction. In general, when dealing with v_i we construct a set V_i of $\Omega(n^{2/3})$ vertices at distance $i_0 - 1$, while avoiding the at most $O(dn^{2/3}) = o(n)$ vertices in $W_1 \cup \cdots \cup W_{i-1}$.

Now fix any other vertex w and construct corresponding sets $\widehat{V}_i, i = 1, 2, \ldots, d/2$ avoiding any vertex in $W_1 \cup \cdots \cup W_{d/2}$. The possible edges between V_i and $\widehat{V}_i$ are unconditioned by this construction and so

$$\mathbf{P}(\exists i : \text{there is no } V_i : \widehat{V}_i \text{ edge}) \leq d(1-p_0)^{\nu_d^2} = O(n^{1/3-\epsilon} e^{-\Omega(dn^{1/3})}) = o(1).$$

We can therefore select $d/2$ edges, one from each $V_i : \widehat{V}_i$, that define internally vertex disjoint paths as required.