Homework 3: Solutions

2.4.14 The expected number of sets of size at most s that contain at least $ks/2$ edges is at most

$$\sum_{t=2k+1}^{s} \binom{n}{t} \binom{\binom{t}{2}}{kt/2} p^{kt/2} \leq \sum_{t=2k+1}^{s} \left(\frac{ne}{t}\right)^t \left(\frac{t^2 e}{kt}\right)^{kt/2} p^{kt/2}$$
$$= \sum_{t=2k+1}^{s} \left(\left(\frac{t}{n}\right)^{k/2-1} \left(\frac{e^{1+2/k}c}{k}\right)^{k/2}\right)^t = o(1)$$

if say, $s \leq s_0 = \theta n$ where $\theta = \frac{1}{2}(e^{1+2/k}c/k)^{-k/(k-2)}n$.

This means that w.h.p. every set of size at most s_0 contains a vertex with fewer than than k neighbors in the set. Thus w.h.p. either the k-core is empty or it has size greater than s_0.

2.4.15 The expected number of vertices X_g on cycles of length at most g is

$$\sum_{k=3}^{g} \binom{n}{k} \frac{k!}{2} \left(\frac{c}{n}\right)^k \leq \sum_{k=3}^{g} \frac{c^k}{2} \leq c^g.$$

So, w.h.p. $X_g \leq c^g \log n$. If $c = 1+\epsilon$ we take $g = 10/\epsilon$ and remove the vertices on short cycles to obtain a graph H. W.h.p. we have a graph with $v = n - O(\log n)$ vertices and girth greater than g. W.h.p., it also has at least

$$(1+o(1))\frac{(1+\epsilon)n}{2} - \frac{X_g \log n}{\log\log n} = (1+o(1))\frac{(1+\epsilon)n}{2} \text{ edges.} \qquad (1)$$

If $G_{n,p}$ is planar then so is H. Suppose that H has v vertices, e edges and f faces. Then we have

$$v - e + f = 2 \text{ and } 2e \geq gf.$$

The first equation is Euler's formula and the inequality follows from the fact that every edges is on exactly 2 faces. So,

$$e \leq v\left(1 - \frac{2}{g}\right)^{-1} \leq (n - o(n))\,(1+3\epsilon)\,10$$

which contradicts (1).

2.4.17 let X_k denote the number of copies of C_k in $G_{n,p}$ and assume for now that $p = \omega/n$ where $\omega = o(\log n)$. Then we have

$$\mathbf{E}(X_k) = \binom{n}{k}\frac{(k-1)!}{2}p^k \sim \frac{\omega^k}{2k} \to \infty.$$

Next, if $Y_{k,t}$ denotes the number of k-cycles in K_n that share t edges with the cycle $(1, 2, \ldots, k, 1)$, then

$$\begin{aligned}
\mathbf{E}(X_k^2) &= \mathbf{E}(X_k) + \mathbf{E}(X_k) \sum_{t=0}^{k-1} Y_{k,t} p^{k-t} \\
&\leq \mathbf{E}(X_k) + \mathbf{E}(X_k)^2 + \sum_{t=1}^{k-1} \binom{k}{t} n^{k-t} p^{k-t} \\
&\leq \mathbf{E}(X_k) + \mathbf{E}(X_k)^2 + 2^k \mathbf{E}(X_k) \sum_{t=1}^{k-1} \omega^{k-t} \\
&= (1 + o(1)) \mathbf{E}(X_k)^2.
\end{aligned}$$

The result follows from the Chebyshev inequality. If ω grows faster than claimed then we use monotonicity.