Homework 2: Solutions

1.4.8 The probability that $G_{n,p} = K_n$ is $p^N \to 0$ and so its diameter will be at least two w.h.p. On the other hand, let $\mathcal{A}_{x,y}$ be the event that there does not exist $z \neq x, y$ such that $\{x, z\}, \{y, z\} \in E(G_{n,p})$. Then,

$$\mathbf{P}(\exists x, y : \mathcal{A}_{x,y}) \leq n^2(1-p)^{n-2} \leq n^2 e^{-(n-2)p} \to 0.$$

1.4.9 Let $Z = |X| = Z_1 + Z_2 + \cdots Z_n$ where $Z_j = 1_{j \in X}$. Then for $j \in [n]$,

$$\mathbf{E}(Z_j) = \mathbf{P}(Z_j = 1) = \left(1 - \frac{1}{n}\right)^n \sim e^{-1}.$$

So,

$$\mathbf{E}(Z) \sim ne^{-1}.$$

We now estimate $\mathbf{E}(Z_i Z_j) = \mathbf{P}(\{i, j\} \subseteq X)$ for $i \neq j$.

$$\mathbf{P}(\{i, j\} \subseteq X) = \left(1 - \frac{2}{n}\right)^n$$

and so

$$\mathbf{E}(Z^2) = n\left(1 - \frac{1}{n}\right)^n + n(n-1)\left(1 - \frac{2}{n}\right)^n.$$

Thus

$$Var(Z) = \mathbf{E}(Z) + \mathbf{E}(Z)^2 \left(\frac{n(n-1)\left(1 - \frac{2}{n}\right)^n}{n^2\left(1 - \frac{1}{n}\right)^{2n}} - -1 \right) \leq \mathbf{E}(Z).$$

Applying the Chebyshev iequality, we have

$$\mathbf{P}(|Z - \mathbf{E}(Z)| \geq \epsilon \mathbf{E}(Z)) \leq \frac{1}{\epsilon^2 \mathbf{E}(Z)}.$$

Putting $\epsilon = n^{-1/4}$ gives us what we want.

2.4.5 Let X denote the number of pairs (e, H) where H is a unicyclic graph with n vertices $[n]$ and n edges and e is an edge of $C(H)$, where $C(H)$ is the unique cycle of H. Then

- $X = n^{n-2}(N - n + 1)$ where $N = \binom{n}{2}$, counting $(e, e + T)$ where T is a spanning tree.
- $X = \sum_{k=1}^{n} kC_k$ where C_k is the number of unicyclic H whose cycle has k edges.

So,

$$\frac{n^{n-2}(N - n + 1)}{C(n, n)} = \frac{\sum_{k=1}^{n} kC_k}{C(n, n)} = \mathbf{E}(|C(H)|).$$