Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 2: Due Monday September 23.

Q1 Formulate the following as an integer program: There are n students and exams $E_1, E_2, \ldots, E_m \subseteq [n]$ need to be scheduled. Student i takes exams $S_i \subseteq [m]$. There are s rooms available and each room can hold r students. The rules are
(i) A student must not be asked to take more than one exam per day;
(ii) Several different exams can be held in the same room provided there is capacity in the room to hold the students.
(iii) No student has to take 3 exams in 3 consecutive days.

The problem is to minimise the number of days needed to carry out all of the exams.
(Hint: let $x_{i,j,k} = 1$ iff exam i takes place in room j on day k and $y_i = 1$ if there is an exam on day i.)

Solution: Define

$$x_{i,j,k} = \begin{cases} 1 & \text{Exam } i \text{ takes place in room } j \text{ on day } k \\ 0 & \text{Otherwise} \end{cases}$$

and

$$y_i = \begin{cases} 1 & \text{At least one exam takes place on day } j \\ 0 & \text{Otherwise} \end{cases}$$

and

$$a_{l,i} = \begin{cases} 1 & l \in S_i, \text{ i.e. student } l \text{ takes exam } i \\ 0 & \text{otherwise} \end{cases}$$

Then we have to solve the problem

$$\text{Minimise } \sum_{i=1}^{m} y_i$$

Subject to

$$\sum_{j=1}^{s}\sum_{k=1}^{n} x_{i,j,k} = 1 \qquad \text{for all } i.$$

$$x_{i,j,k} \leq y_k \qquad \text{for all } i, j, k.$$

$$\sum_{i=1}^{m} |E_i| x_{i,j,k} \leq r \qquad \text{for all } j, k.$$

$$\sum_{i=1}^{m}\sum_{j=1}^{s} a_{l,i} x_{i,j,k} \leq 1 \text{ for all } l, k$$

$$\sum_{i=1}^{m}\sum_{j=1}^{s} a_{l,i}(x_{i,j,k} + x_{i,j,k+1} + x_{i,j,k+2}) \leq 2 \text{ for all } l, k$$

Q2 The government has asked for and received bids on m construction projects from each of n firms. No firm will be awarded more than one contract and for political reasons no more than p large contracts are to go to foreign firms. Projects $1, 2, \ldots, \ell$ are large and firms $1, 2, \ldots, f$ are foreign. If $b_{i,j}$ is the bid by firm i for project j, which bids should be accepted to minimise the total cost?

Solution:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} x_{ij}$$

$$\begin{aligned} s.t. & \\ \sum_{i=1}^{n} x_{ij} &= 1 \quad \forall j \\ \sum_{j=1}^{m} x_{ij} &\leq 1 \quad \forall i \\ \sum_{i=1}^{f}\sum_{j=1}^{l} x_{ij} &\leq p \\ x_{ij} &\in \{0, 1\} \end{aligned}$$

The first constraint ensures that each project is assigned to some firm. The second that no firm gets more than one project and the third that the foreign firms get less than p.

Q3 Solve the following problem by a cutting plane algorithm:

$$\begin{array}{llllll} \text{minimise} & 4x_1 & + & 5x_2 & +3x_3 & \\ \text{subject to} & & & & & \\ & 2x_1 & + & x_2 & -x_3 & \geq 2 \\ & x_1 & + & 4x_2 & +x_3 & \geq 13 \end{array}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution:

Initial tableau

x_1	x_2	x_3	x_4	x_5	R.H.S	
-4	-5	-3	0	0	0	z
-2	-1	1	1	0	-2	x_4
-1	-4	-1	0	1	-13	x_5 ←
	↑					

x_1	x_2	x_3	x_4	x_5	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	$\frac{13}{4}$	x_2

Primal feasible, but the solution is not integral.
We add a cut which eliminates the current optimal solution.

$$\frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{3}{4}x_5 - y_1 = \frac{1}{4}$$

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	0	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	0	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	0	$\frac{13}{4}$	x_2
$\frac{-11}{4}$	0	$\frac{-1}{4}$	0	$\frac{-3}{4}$	+1	$\frac{-1}{4}$	y_1 ←
				↑			

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	$\frac{-5}{3}$	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	$\frac{1}{3}$	x_5

The solution is primal feasible and so optimal but still not integer. We add a cut which eliminates the current optimal solution.

$$\frac{-1}{3}x_1 - \frac{1}{3}x_3 + y_2 = \frac{1}{3}$$

We obtain tableau

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	0	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	0	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	x_5
$\frac{-1}{3}$	0	$\frac{-1}{3}$	0	0	1	$\frac{1}{3}$	$y_2 \leftarrow$
		$\uparrow$					

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
-1	0	0	0	0	-4	18	z
-3	0	0	1	0	4	0	x_4
0	1	0	0	0	1	3	x_2
0	0	0	0	1	1	0	x_5
1	0	1	0	0	-3	1	x_3

Which is optimal integral.