Department of Mathematical Sciences
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OPERATIONS RESEARCH II 21-393

Answers to homework 1.

Describe a Dynamic programming solution to the following problems:

Q1 Solve the following knapsack problem:

$$\begin{array}{lccccccc} \text{maximise} & 2x_1 & + & 4x_2 & + & 10x_3 & & \\ \text{subject to} & & & & & & & \\ & 2x_1 & + & 3x_2 & + & 6x_3 & \leq & 15 \end{array}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution:

w	f_1	δ_1	f_2	δ_2	f_3	δ_3
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	2	1	2	0	2	0
3	2	1	2	0	2	0
4	4	1	4	0,1	4	0
5	4	1	6	1	6	0
6	6	1	8	1	10	1
7	6	1	8	1	10	1
8	8	1	10	1	12	1
9	8	1	12	1	14	1
10	10	1	12	1	14	1
11	10	1	14	1	16	1
12	12	1	16	1	20	1
13	12	1	16	1	20	1
14	14	1	18	1	22	1
15	14	1	20	1	24	1

Solution: $x_1 = 0, x_2 = 1, x_3 = 2$. Maximum = 24.

Q2 Consider a 2-D map with a horizontal river passing through its center. There are n cities on the southern bank with x-coordinates $a(1)...a(n)$ and n cities on the northern bank with x-coordinates $b(1)...b(n)$. You want to connect as many north-south pairs of cities as possible with bridges such that no two bridges cross. When connecting cities, you can only connect city i on the northern bank to city i on the southern bank. Construct a Dynamic Programming solution to this problem. (You can assume that $a(1) < a(2) < \cdots < a(n)$, but you **cannot** assume that $b(1) < b(2) < \cdots < b(n)$. If both sequences are increasng, then the problem is trivial).

Solution: Let $f(j)$ be the maximum number of bridges choosable if we only use $(a(i), b(i),\ i \geq j)$. Then

$$f(j) = \max \begin{cases} f(j+1) & \text{do not choose } (a(j), b(j)) \\ 1 + f(\min\{k > j :\ b(k) > b(j)\}) & \text{choose } (a(j), b(j)) \end{cases}.$$

Q3 A company manufactures two products A and B at a certain facility. The demands for the products are $a_i, b_i, i = 1, 2, \ldots, n$ over the next n periods. The cost of making x of either product is $c(x)$ and there is room to store H in total of the two products. Cleaning problems require that only one product can be manufactured in any one period and that the same product cannot be manufactured for more than 3 consecutive periods. Assume that at the beginning of period one there is $H/2$ of each product in storage. The problem is to minimise total cost, given that all demands must be met.

Solution: Let $f_r(\delta_1, \delta_2, \delta_3, i_A, i_B)$ denote the minimum cost of meeting demand over periods $r, r+1, \ldots, n$ given that we manufactured product A in period $r - i$ if $\delta_{4-i} = 1$. If $r \leq 4$ then we drop some of these δ_i parameters. In general we have

$$f_r(\delta_1, \delta_2, \delta_3, i_A, i_B) = \min_x \{c(x) +$$
$$\min\{f_{r+1}(\delta_2, \delta_3, 0, i_A - a_r, i_B + x - b_r), f_{r+1}(\delta_2, \delta_3, 1, i_A + x - a_r, i_B - b_r)\}\}$$
$$\text{if not all } \delta_i \text{ are equal.}$$

$$f_r(0, 0, 0, i_A, i_B) = \min_x \{c(x) + f_{r+1}(0, 0, 1, i_A + x - a_r, i_B - b_r)\}.$$
$$f_r(1, 1, 1, i_A, i_B) = \min_x \{c(x) + f_{r+1}(1, 1, 1, i_A - a_r, i_B + x - b_r)\}.$$