Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 2: Due Monday October 9.

Q1 Solve the following 2-person zero-sum games:

$$\begin{bmatrix} 6 & 2 & 4 \\ 5 & 2 & 5 \\ 4 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -2 & -2 \\ 4 & 1 & 0 & -2 & -3 \end{bmatrix}$$

Solution (2,2) is a saddle point for the first game. Thus the solution is for player A to use 1 and player B to use 2. The value of the game is 2.

For the second game we have the following sequence of row/column removals because of domination:

$$\text{Remove column strategy 1.} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & -2 & -2 \\ 1 & 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 2.} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\text{Remove column strategy 3.} \begin{bmatrix} 0 & -1 \\ 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}.$$

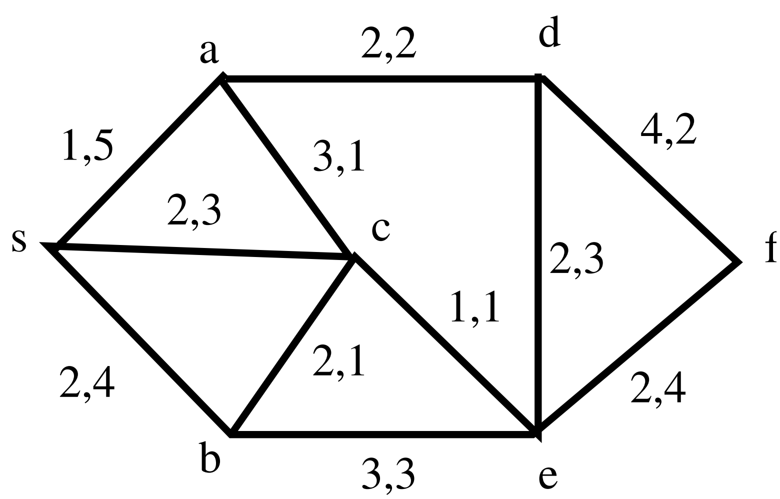
Remove row strategy 1. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -2 \\ -2 & -3 \end{bmatrix}$

Remove row strategy 4. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -2 & -3 \end{bmatrix}$

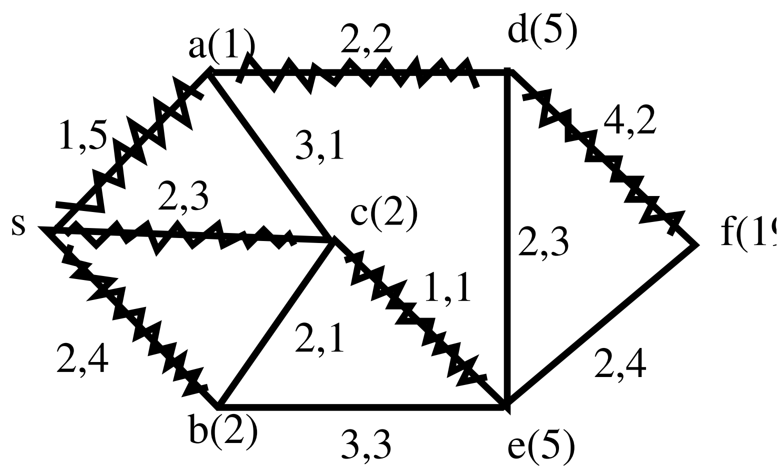
Remove row strategy 5. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The optimal strategies for this game are for player A to play rows 2 and 3 with probability 1/2 each. Similarly, player B plays columns 4 and 5 with probabilty 1/2 each.

Q2 Find a shortest path from s to all other nodes in the digraph below. Each edge (x, y) is labelled by a pair (a, b) and the length of the corresponding arc is $a + bt$ where t is the time the path reaches x. All arcs are directed lexicograhically e.g. (c, e) is directed from c to e.



Solution



Q3 There are two machines available for the processing of $n = 2m$ jobs. The processing time of job j is $p_j > 0$ for $j = 1, 2, \ldots, n$. The objective is to assign jobs to machines in order to minimise $\sum_{j=1}^{n} C_j$ where C_j is the completion time of job j.

(a) Suppose that in an optimum schedule machine 1 processes jobs $i_1, i_2, \ldots, i_s$ and machine 2 processes jobs $j_1, j_2, \ldots, j_t$ in this order. Show that the contribution of machine 1 to the objective function is

$$sp_{i_1} + (s-1)p_{i_2} + \cdots + 2p_{i_{s-1}} + p_{i_s}.$$

(b) Show that $p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_s}$.

(c) Show that $s = t = m$ in the optimal solution.
(Hint: if $s \geq m + 1$, see the effect of moving job i_1 to the front of machine 2's list.)

(d) Show that $p_{i_m} \geq p_{j_{m-1}}$.

Deduce the structure of an optimal solution.

Solution:

(a) The completion time of job i_t is $p_{i_1} + \cdots + p_{i_t}$ and so the contribution of Machine 1 is

$$p_{i_1} + (p_{i_1} + p_{i_2}) + (p_{i_1} + p_{i_2} + p_{i_3}) + \cdots = sp_{i_1} + (s-1)p_{i_2} + \cdots + 2p_{i_{s-1}} + p_{i_s}.$$

(b) If $p_{i_k} > p_{i_{k+1}}$ then interchanging them changes the contribution in (a) by

$$((s-k+1)p_{i_{k+1}} + (s-k)p_{i_k}) - ((s-k+1)p_{i_k} + (s-k)p_{i_{k+1}})$$
$$= p_{i_{k+1}} - p_{i_k} < 0.$$

(c) Suppose that we do as in the hint. Then the change in objective is $(t+1-s)p_{i_1} < 0$.

(d) If $p_{i_m} < p_{j_{m-1}}$ then we can swap them over. The total change in cost will be $p_{i_m} - p_{j_{m-1}} < 0$.

Given (a) – (d) we see that we should first identify the two jobs with the longest processing times and assign one machine one and set them aside. Then we should recursively solve the problem on the reminaining $n-2$ jobs and then add each of these two jobs to a different machine. This leads to the following algorithhm: (i) First relabel the jobs so that $p_1 \leq p_2 \leq \cdots \leq p_n$. Then put jobs 1,3,5,... on machine one and the rest on machine 2.