Conditional Probability

Suppose $A \subseteq \Omega$. We define an *induced* probability $\mathbf{P}_A$ by

$$\mathbf{P}_A(\omega) = \frac{\mathbf{P}(\omega)}{\mathbf{P}(A)} \qquad \text{for } \omega \in A.$$

Usually write $\mathbf{P}(B \mid A)$ for $P_A(B)$.

If B is an arbitrary subset of Ω we write

$$\mathbf{P}(B \mid A) = \mathbf{P}_A(A \cap B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)}.$$

Two fair dice are thrown. Given that the first shows 3, what is the probability that the total exceeds 6? The answer is obviously $\frac{1}{2}$ since the second must show 4,5 or 6.

In detail: $\Omega = [6]^2$ with uniform mesaure. Let

$$\begin{aligned} A &= \{(i,j) \in [6]^2 : \ i = 3\}. \\ B &= \{(i,j) \in [6]^2 : \ i + j > 6\} \\ A \cap B &= \{(i,j) \in [6]^2 : \ i = 3, j > 3\}. \end{aligned}$$

Thus

$$\mathbf{P}(A) = \frac{1}{6}, \ \mathbf{P}(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

and so

$$\mathbf{P}(A \mid B) = \frac{1/12}{1/6} = \frac{1}{2}.$$

Suppose that a family has two children and that each child is equally likely to be a Boy or a Girl. What is the probability that the family has two boys, given that it has at least one Boy.

Probability Space: {BB,BG,GB,GG}
with uniform measure, where GB means that the first child is a Girl and the second child is a Boy etc..

$\mathcal{A} = \{$At least one child is a Boy$\}$=$\{BG, GB, BB\}$.
So

$$\mathbf{P}(\mathcal{A}) = 3/4.$$

and

$$\mathbf{P}(\{BB\} \mid \mathcal{A}) = \frac{\mathbf{P}(\{BB\} \cap \mathcal{A})}{\mathbf{P}(\mathcal{A})} = \frac{\mathbf{P}(\{BB\})}{\mathbf{P}(\mathcal{A})} = \frac{1}{3}.$$

Monty Hall Paradox

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say Number a, and the host, who knows what's behind the other doors, opens another door b which has a goat. He then says to you, 'Do you want to pick door Number $c \neq a, b$?' Is it to your advantage to take the switch?

The door hiding the car has been chosen randomly, if door a hides the car then the host chooses b randomly and there is an implicit assumption that you prefer a car to a goat.

Incorrect Analysis:

Probability space is $\{1, 2, 3\}$ where i denotes that the car is behind door i and $\mathbf{P}(i) = \frac{1}{3}$ for $i = 1, 2, 3$.

Answer: Assume for example that $a = 1$. Then for $b = 2, 3$,

$$\mathbf{P}(1 \mid \text{not } b) = \frac{\mathbf{P}(1)}{\mathbf{P}(\text{not } b)} = \frac{\mathbf{P}(1)}{\mathbf{P}(1) + \mathbf{P}(5 - b)} = \frac{1}{2}.$$

So there is no advantage to be gained from switching.

Correct Analysis:

Assume that $a = 1$.
Probability space is $\{12, 13, 23, 32\}$ where ij denotes that the car is behind door i and the host opens door j.

$\mathbf{P}(12) = \mathbf{P}(13) = \frac{1}{6}$ and $\mathbf{P}(23) = \mathbf{P}(32) = \frac{1}{3}$.

So,

$$\mathbf{P}(1) = \mathbf{P}(12) + \mathbf{P}(13) = \frac{1}{3}.$$

So there is an advantage to be gained from switching.

Look at it this way: The probability the car is not behind door a is 2/3 and switching causes you to win whenever this happens!

Binomial n coin tosses.
$p = \mathbf{P}(Heads)$ for each toss.
$\Omega = \{H, T\}^n$.

$$\mathbf{P}(\omega) = p^k(1-p)^{n-k}$$

where k is the number of H's in ω.
E.g. $\mathbf{P}(HHTTHTHHTHHTHT) = p^8(1-p)^6$.

Fix k. $A = \{\omega : \ H$ appears k times$\}$
$\mathbf{P}(A) = \binom{n}{k} p^k (1-p)^{n-k}$.
If $\omega \in A$ then

$$\mathbf{P}_A(\omega) = \frac{p^k(1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

i.e. *conditional* on there being k heads, each sequence with k heads is equally likely.

Balls in boxes

m distinguishable balls in n distinguishable boxes.

Let

$$E_i = \{\text{Box } i \text{ is empty}\}.$$

$$\Pr(E_1) = \Pr(E_2) = \left(1 - \frac{1}{n}\right)^m$$

and

$$\Pr(E_1 \cap E_2) = \left(1 - \frac{2}{n}\right)^m < \Pr(E_1)\Pr(E_2).$$

So

$$\Pr(E_1 \mid E_2) < \Pr(E_1).$$

We say that the two events are *negatively correlated*.

Law of Total Probability

Let $B_1, B_2, \ldots, B_n$ be pairwise disjoint events which partition Ω. For any other event A,

$$\mathbf{P}(A) = \sum_{i=1}^{n} \mathbf{P}(A \mid B_i)\mathbf{P}(B_i).$$

Proof

$$\begin{aligned}
\sum_{i=1}^{n} \mathbf{P}(A \mid B_i)\mathbf{P}(B_i) &= \sum_{i=1}^{n} \mathbf{P}(B_i \cap A) \\
&= \mathbf{P}\Big(\bigcup_{i=1}^{n} (B_i \cap A)\Big) \quad (1) \\
&= \mathbf{P}(A).
\end{aligned}$$

There is equality in (1) because the events $B_i \cap A$ are pairwise disjoint.

Suppose that we have 2 crooked dice and that if the outcome of the first is X then the outocme of the second Y satisfies
Y is uniformly chosen from $\{X-1, X, X+1\}$.
(If $X = 1$ or 6 then Y has 2 equally likely values.)
What is the probability that $X = Y$, assuming that $\Pr(X = i) = \frac{1}{6}$ for $i = 1, 2, \ldots, 6$?

$$\begin{aligned}
\Pr(X = Y) &= \sum_{i=1}^{6} \Pr(X = Y \mid X = i) \Pr(X = i) \\
&= \frac{1}{6} \sum_{i=1}^{6} \Pr(X = Y \mid X = i) \\
&= \frac{1}{6} \left(\frac{1}{2} + \frac{1}{3} + + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} \right) \\
&= \frac{7}{18}.
\end{aligned}$$

We are given two urns, each containing a collection of coloured balls. Urn 1 contains 2 Red and 3 Blue balls. Urn 2 contains 3 Red and 4 Blue balls. A ball b_1 is drawn at random from Urn 1 and placed in Urn 2 and then a ball b_2 is chosen at random from Urn 2 and examined. What is the probability that b_2 is Blue?

Let

$$\begin{aligned} A &= \{b_2 \text{ is Blue}\} \\ B_1 &= \{b_1 \text{ is Blue}\} \\ B_2 &= \{b_1 \text{ is Red}\} \end{aligned}$$

Then

$$\mathbf{P}(B_1) = \frac{3}{5},\ \mathbf{P}(B_2) = \frac{2}{5}$$

$$\mathbf{P}(A \mid B_1) = \frac{5}{8},\ \mathbf{P}(A \mid B_2) = \frac{1}{2}.$$

So,

$$\mathbf{P}(A) = \frac{5}{8} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{23}{40}.$$

2 sets $S, T \subseteq [n]$ are chosen (i) independently and (ii) uniformly at random from all possible sets. ($\Omega = \{0,1\}^{2n}$). Let

$$A = \{|S| = |T| \text{ and } S \cap T = \emptyset\}.$$

For each $X \subseteq [n]$ we let $B_X = \{S = X\} = \{(X,T) : T \subseteq [n]\}$. Thus for each X, $\mathbf{P}(B_X) = 2^{-n}$. So,

$$\begin{aligned} \mathbf{P}(A) &= \sum_X \mathbf{P}(A \mid B_X)\mathbf{P}(B_X) \\ &= 2^{-n} \sum_X \binom{n-|X|}{|X|} 2^{-n} \qquad (2) \\ &= 4^{-n} \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k}. \end{aligned}$$

(2) follows from the fact that there are $\binom{n-|X|}{|X|}$ subsets of the same size as X which are disjoint from X.

Independence

Two events A, B are said to be *independent* if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B),$$

or equivalently

$$\mathbf{P}(A \mid B) = \mathbf{P}(A).$$

(i) **Two Dice**

$A = \{\omega : x_1 \text{ is odd}\}$, $B = \{\omega : x_1 = x_2\}$.

$|A|$=18, $|B|$=6, $|A \cap B|$=3.

$\mathbf{P}(A) = 1/2$, $\mathbf{P}(B) = 1/6$, $\mathbf{P}(A \cap B) = 1/12$.

A, B are independent.

(ii) $A = \{x_1 \geq 3\}$, $B = \{x_1 \geq x_2\}$.

$|A|$=24, $|B|$=21, $|A \cap B|$=18.

$\mathbf{P}(A) = 2/3$, $\mathbf{P}(B) = 7/12$, $\mathbf{P}(A \cap B) = 1/2$.

A, B are not independent.

Random Bits

Suppose $\Omega = \{0,1\}^n = \{(x_1, x_2, \ldots, x_n) : x_j = 0/1\}$ with uniform distribution.
Suppose event A is determined by the values of x_i, $i \in \Delta_A$
e.g. if $A = \{x_1 = x_2 = \cdots = x_{10} = 0\}$ then $\Delta_A = \{1, 2, \ldots, 10\}$.

More Precisely: for $S \subseteq [n]$ and $x \in \Omega$ let $x_S \in \{0,1\}^S$ be defined by $(x_S)_i = x_i$, $i \in S$.
Ex. $n = 10$, $S = \{2, 5, 8\}$ and
$x = (0, 0, 1, 0, 0, 1, 1, 1, 1, 0\}$. $x_S = \{0, 0, 1\}$.

A is *determined* by Δ_A if $\exists S_A \subseteq \{0,1\}^{\Delta_A}$ such that $x \in A$ iff $x_{\Delta_A} \in S_A$. Furthermore, no subset of Δ_A has this property.

In our example above,
$S_A = \{(0,0,0,0,0,0,0,0,0,0)\}$ – ($|S_A| = 1$ here.)

Claim: if events A, B are such that $\Delta_A \cap \Delta_B = \emptyset$ then A and B are independent.

$$\mathbf{P}(A) = \frac{|S_A|}{2^{|\Delta_A|}} \text{ and } \mathbf{P}(B) = \frac{|S_B|}{2^{|\Delta_B|}}.$$

$$\begin{aligned}
\mathbf{P}(A \cap B) &= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} 1_{\{x_{\Delta_A} \in S_A, x_{\Delta_B} \in S_B\}} \\
&= \frac{1}{2^n} |S_A|\ |S_B| 2^{n-|I_A|-|I_B|} \\
&= \mathbf{P}(A)\mathbf{P}(B).
\end{aligned}$$