Grid path problems

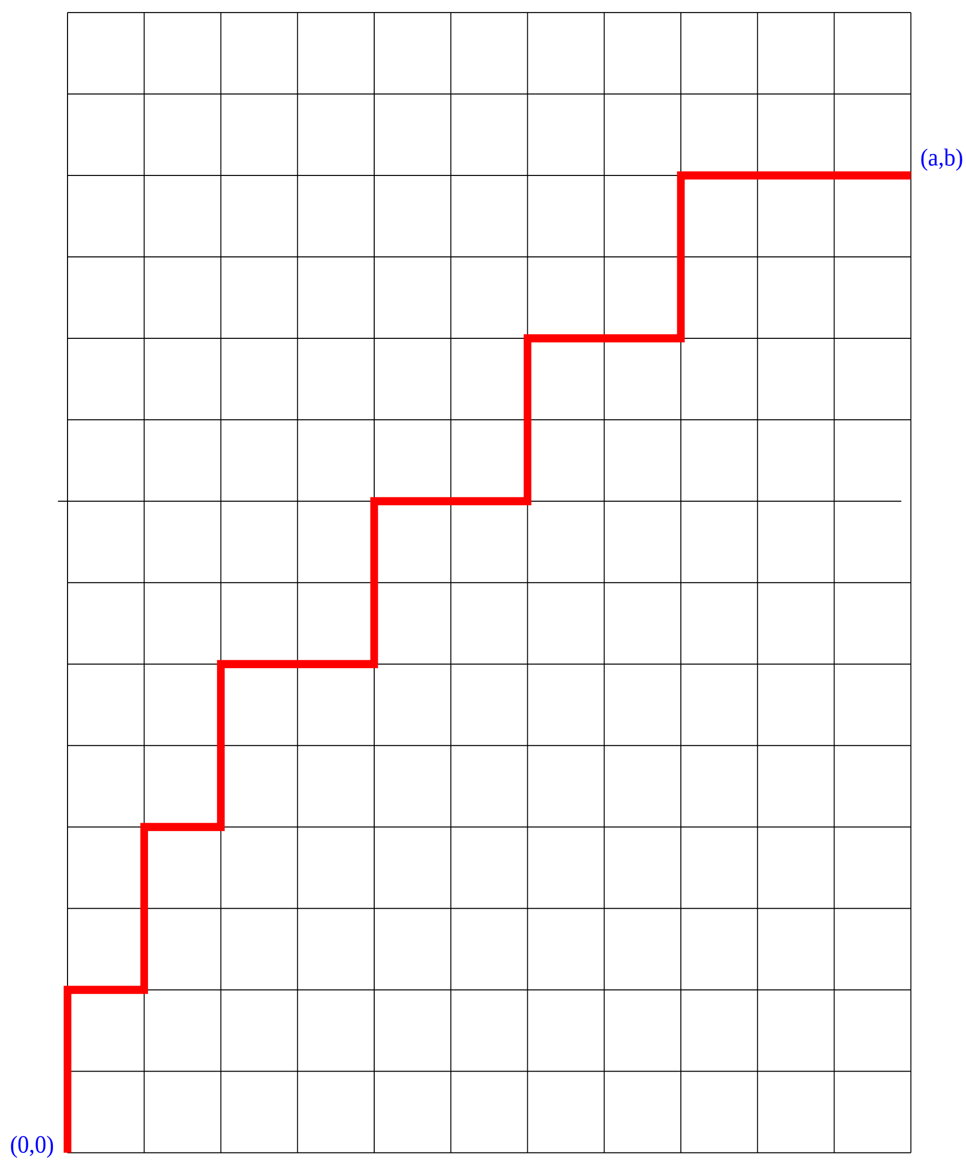
A *monotone path* is made up of segments $(x, y) \to (x + 1, y)$ or $(x, y) \to (x, y + 1)$.

PATHS$((a, b) \to (c, d))$= {monotone paths from (a, b) to (c, d)}.
We drop the $(a, b) \to$ for paths starting at (0,0).
We consider 3 questions: Assume $a, b \geq 0$.

1. How large is PATHS(a, b)?

2. Assume $a < b$. Let PATHS$_{>}(a, b)$ be the set of paths in PATHS(a, b) which do not touch the line $x = y$ except at (0,0). How big is PATHS$_{>}(a, b)$?

3. Assume $a \leq b$. Let PATHS$_{\geq}(a, b)$ be the set of paths in PATHS(a, b) which do not pass through points with $x > y$. How big is PATHS$_{\geq}(a, b)$?


(a,b)
(0,0)

1. $\text{STRINGS}(a, b) = \{x \in \{\text{R}, \text{U}\}^* : \ x$ has a R's and b U's$\}$. *

Natural bijection between $\text{PATHS}(a, b)$ and $\text{STRINGS}(a, b)$:
Path moves to Right, add R to sequence.
Path goes up, add U to sequence.

So

$$|\text{PATHS}(a, b)| = |\text{STRINGS}(a, b)| = \binom{a+b}{a}$$

since to define a string we have state which of the $a + b$ places contains an R.

*$\{\text{R}, \text{U}\}^* =$ set of strings of R's and U's

2. Every path in $\mathsf{PATHS}_{>}(a,b)$ goes through (0,1). So

$$|\mathsf{PATHS}_{>}(a,b)| = |\mathsf{PATHS}((0,1) \to (a,b))| - |\mathsf{PATHS}_{\not>}((0,1) \to (a,b))|.$$

Now

$$|\mathsf{PATHS}((0,1) \to (a,b))| = \binom{a+b-1}{a}$$

and

$$\mathsf{PATHS}_{\not>}((0,1) \to (a,b))| = |\mathsf{PATHS}((1,0) \to (a,b))| = \binom{a+b-1}{a-1}.$$

We explain the first equality momentarily. Thus

$$\begin{aligned} |\mathsf{PATHS}_{>}(a,b)| &= \binom{a+b-1}{a} - \binom{a+b-1}{a-1} \\ &= \frac{b-a}{a+b}\binom{a+b}{a}. \end{aligned}$$

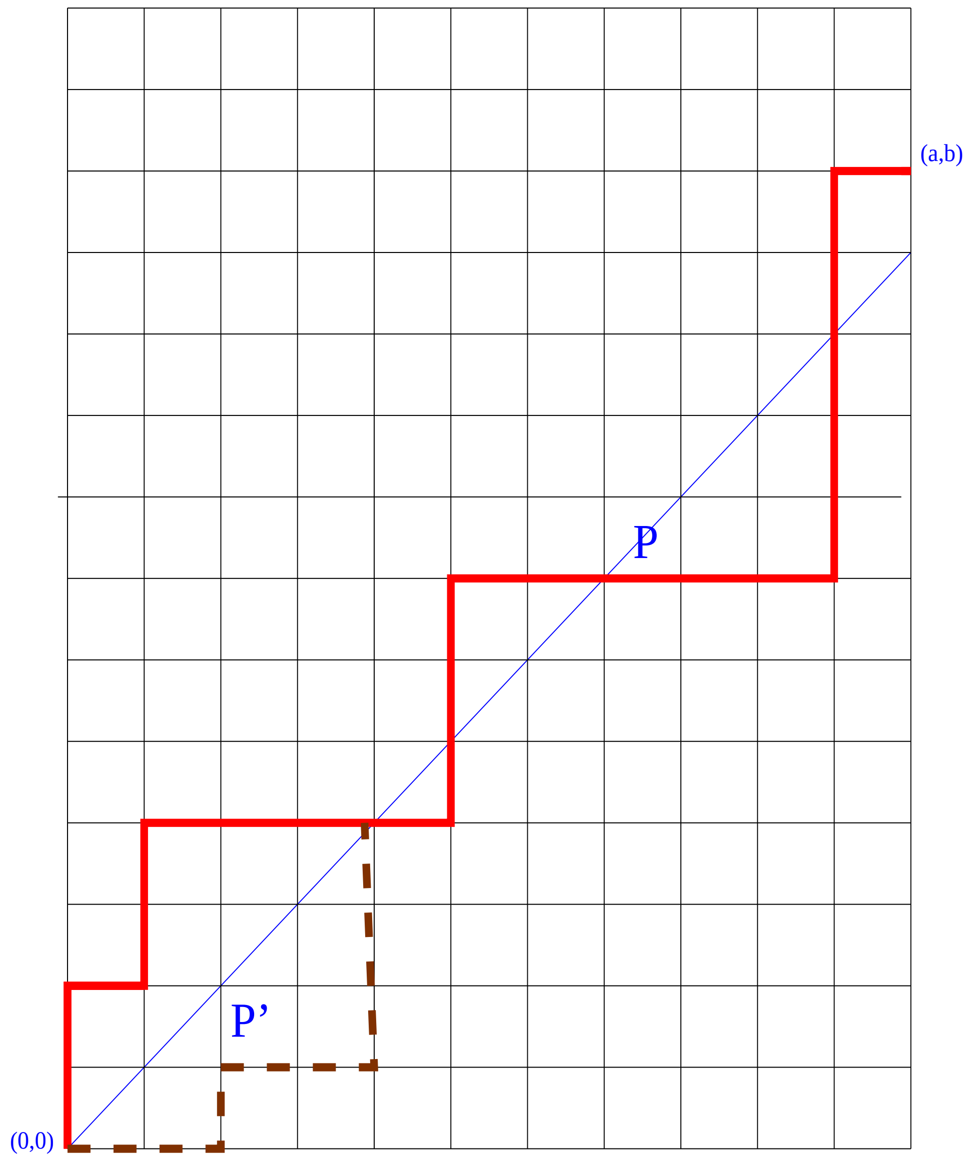
Suppose $P \in \mathsf{PATHS}_{\not\geq}((0,1) \to (a,b))$. We define $P' \in \mathsf{PATHS}((1,0) \to (a,b))$ in such a way that

$$P \to P' \text{ is a bijection.}$$

Let (c,c) be the first point of P, which lies on the line $L = \{x = y\}$ and let S denote the initial segment of P going from $(0,1)$ to (c,c).

P' is obtained from P by deleting S and replacing it by its reflection S' in L.

To show that this defines a bijection, observe that if $P' \in \mathsf{PATHS}((1,0) \to (a,b))$ then a similarly defined *reverse reflection* yields a $P \in \mathsf{PATHS}_{\not\geq}((0,1) \to (a,b))$.


(a,b)
P
P'
(0,0)

3. Suppose $P \in \text{PATHS}_{\geq}(a,b)$. We define $P" \in \text{PATHS}_{>}(a,b+1)$ in such a way that

$$P \rightarrow P" \text{ is a bijection.}$$

Thus

$$|\text{PATHS}_{\geq}(a,b)| = \frac{b-a+1}{a+b+1}\binom{a+b+1}{a}.$$

In particular

$$\begin{aligned} |\text{PATHS}_{\geq}(a,a)| &= \frac{1}{2a+1}\binom{2a+1}{a} \\ &= \frac{1}{a+1}\binom{2a}{a}. \end{aligned}$$

The final expression is called a *Catalan Number*.

The bijection

Given P we obtain P'' by *raising it vertically one position and then adding the segment* $(0,0) \to (0,1)$.

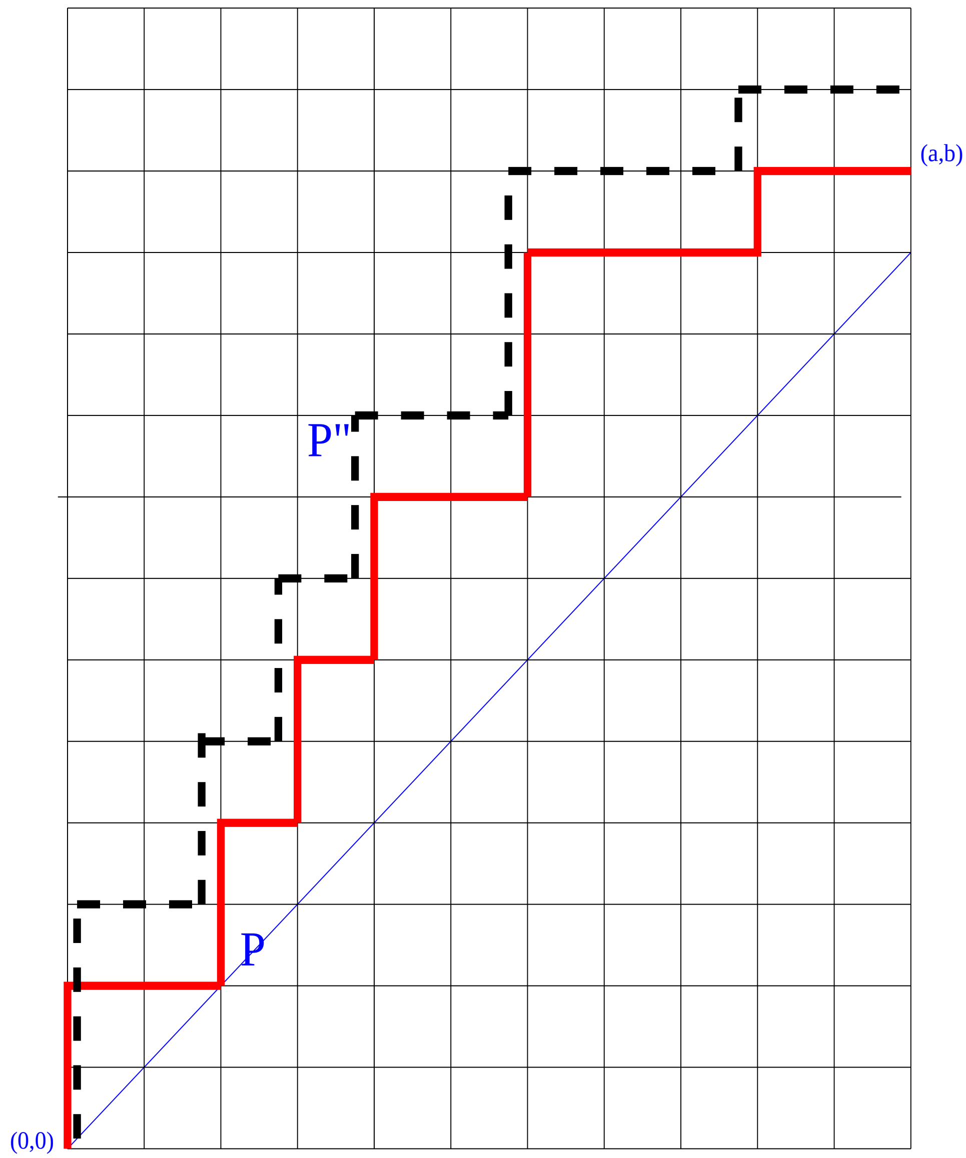
More precisely, if

$$P = (0,0), (x_1, y_1), (x_2, y_2), \ldots, (a, b)$$

then

$$P'' = (0,0), (0,1), (x_1, y_1 + 1), \ldots, (a, b + 1).$$

This is clearly a 1-1 onto function between $\text{PATHS}_{\geq}(a,b)$ and $\text{PATHS}_{>}(a,b+1)$.


(a,b)
P''
P
(0,0)