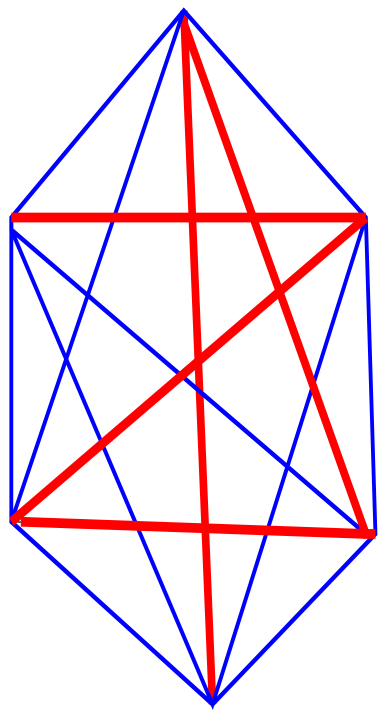
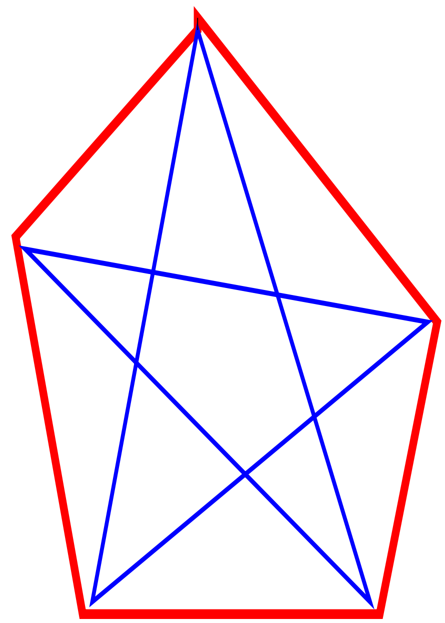
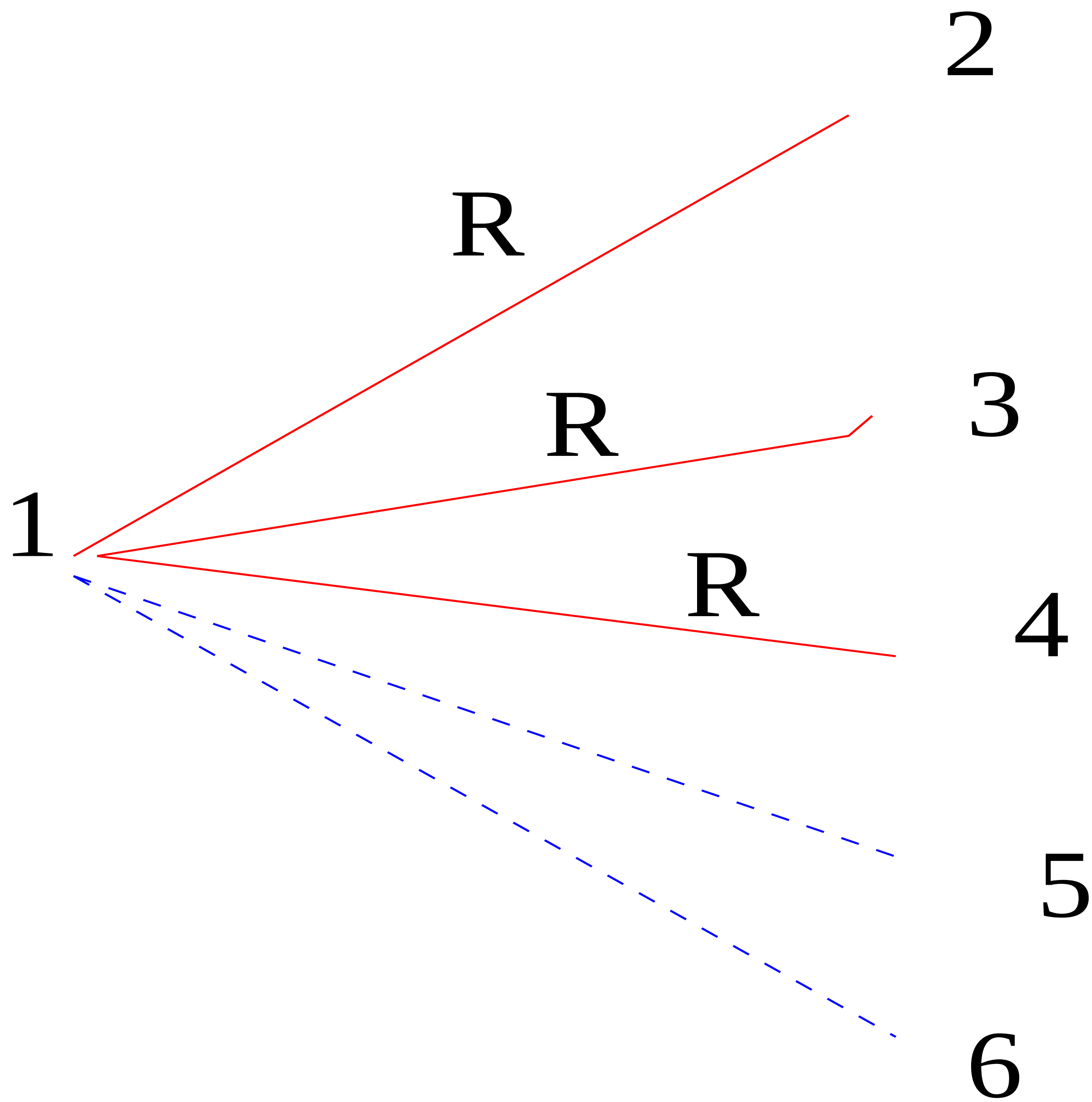
Ramsey's Theorem

Suppose we 2-colour the edges K_6 of Red and Blue. There *must* be either a Red triangle or a Blue triangle.





This is not true for K_5.



There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter implies (1,2,3) is a Red triangle.

Ramsey's Theorem

For all positive integers k, ℓ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of K_N are coloured Red or Blue then then either there is a "Red k-clique" or there is a "Blue ℓ-clique.

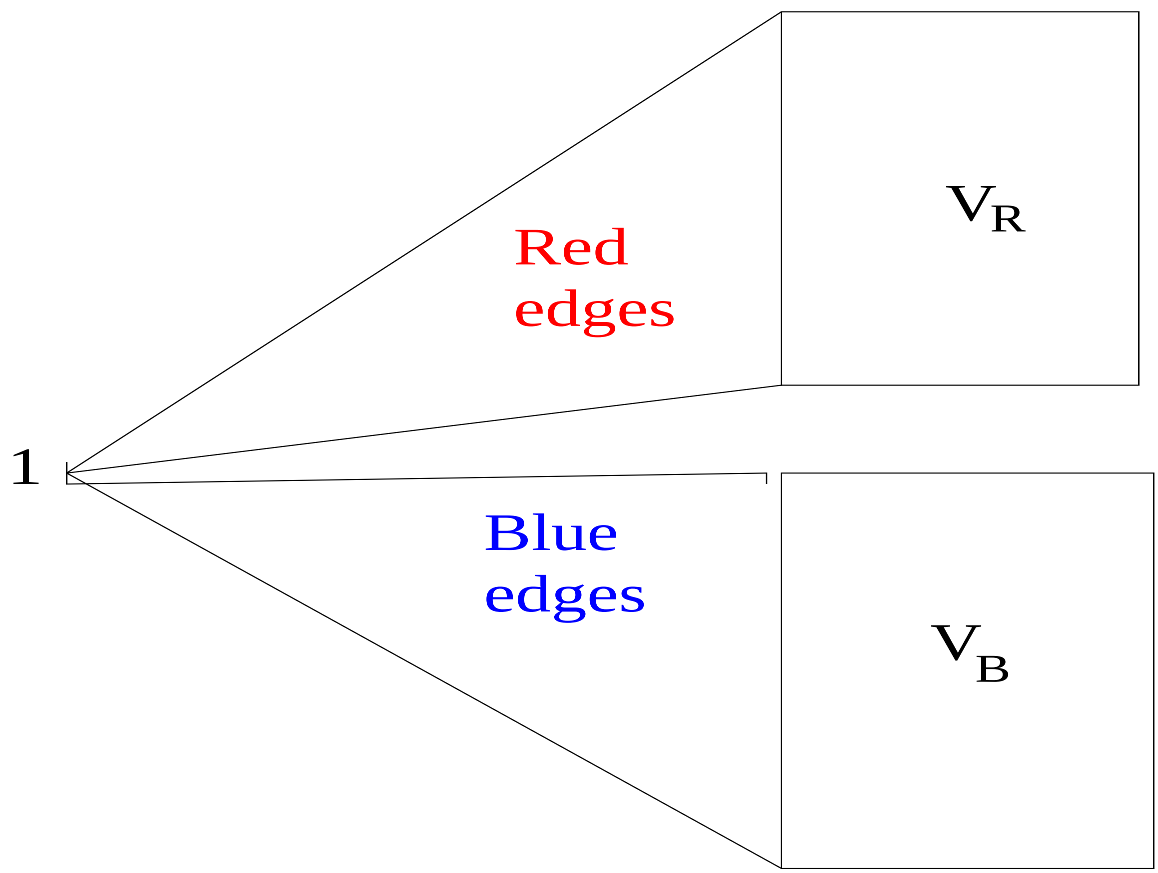
A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

$$\begin{aligned} R(1,k) &= R(k,1) &= 1 \\ R(2,k) &= R(k,2) &= k \end{aligned}$$

Theorem 1

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell).$$

Proof Let $N = R(k,\ell-1) + R(k-1,\ell)$.



$V_R = \{(x :\ (1,x)$ is coloured Red$\}$ and $V_B = \{(x :$ $(1,x)$ is coloured Blue$\}$.

$$|V_R| \geq R(k-1,\ell) \text{ or } |V_B| \geq R(k,\ell-1).$$

Since

$$\begin{aligned} |V_R| + |V_B| &= N-1 \\ &= R(k,\ell-1) + R(k-1,\ell) - 1. \end{aligned}$$

Suppose for example that $|V_R| \geq R(k-1,\ell)$. Then either V_R contains a Blue ℓ-clique – done, or it contains a Red $k-1$-clique K. But then $K \cup \{1\}$ is a Red k-clique.

Similarly, if $|V_B| \geq R(k,\ell-1)$ then either V_B contains a Red k-clique – done, or it contains a Blue $\ell-1$-clique L and then $L \cup \{1\}$ is a Blue ℓ-clique. $\square$

Theorem 2

$$R(k,\ell) \leq \binom{k+\ell-2}{k-1}.$$

Proof Induction on $k+\ell$. True for $k+\ell \leq 5$ say. Then

$$\begin{aligned} R(k,\ell) &\leq R(k,\ell-1)+R(k-1,\ell) \\ &\leq \binom{k+\ell-3}{k-1}+\binom{k+\ell-3}{k-2} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

$\square$

So, for example,

$$\begin{aligned} R(k,k) &\leq \binom{2k-2}{k-1} \\ &\leq 4^k \end{aligned}$$

Theorem 3

$$R(k,k) > 2^{k/2}$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.

We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: $\{$There is a Red k-clique$\}$ and
$\mathcal{E}_B$ be the event: $\{$There is a Blue k-clique$\}$.

We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.
Let $\mathcal{E}_{R,j}$ be the event: $\{C_j$ is Red$\}$.
Now

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) \\
&= 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \\
&\leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
$$