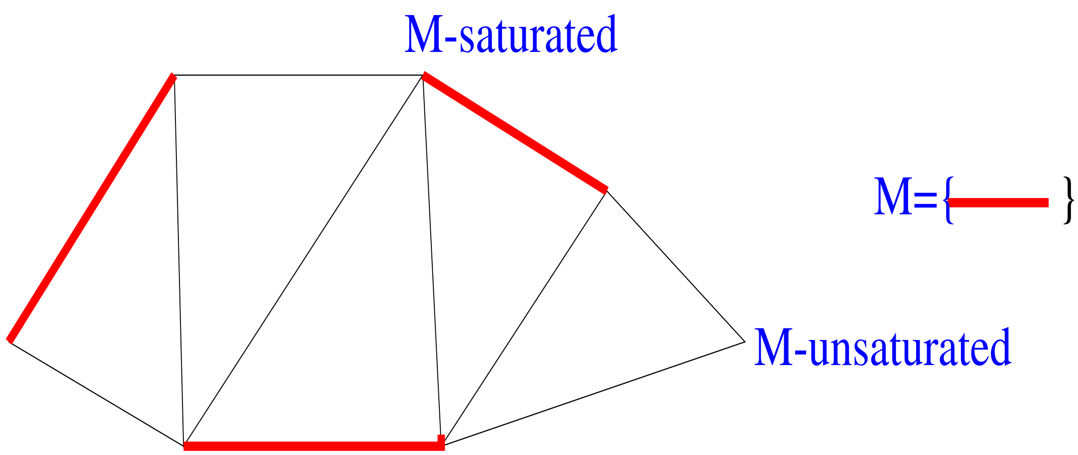
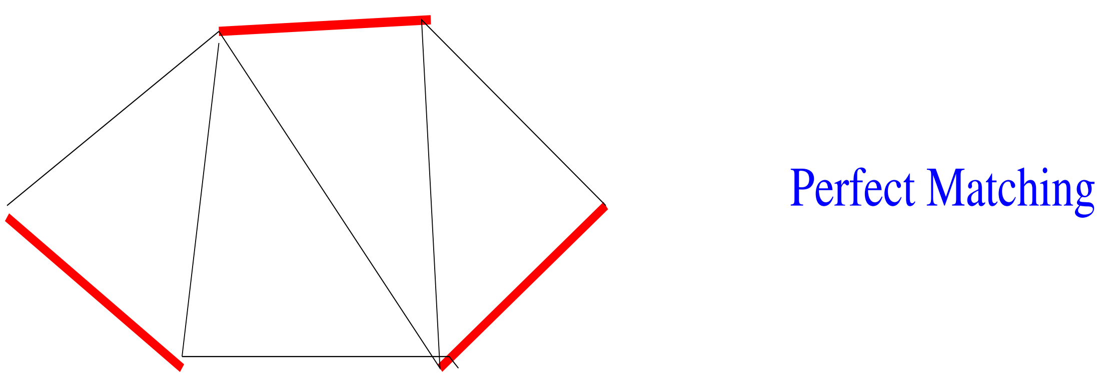
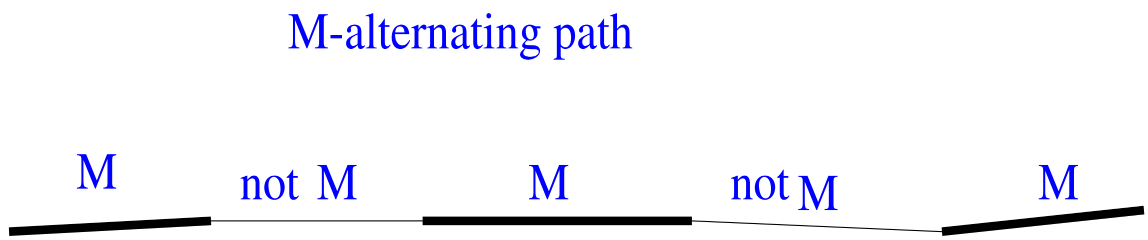
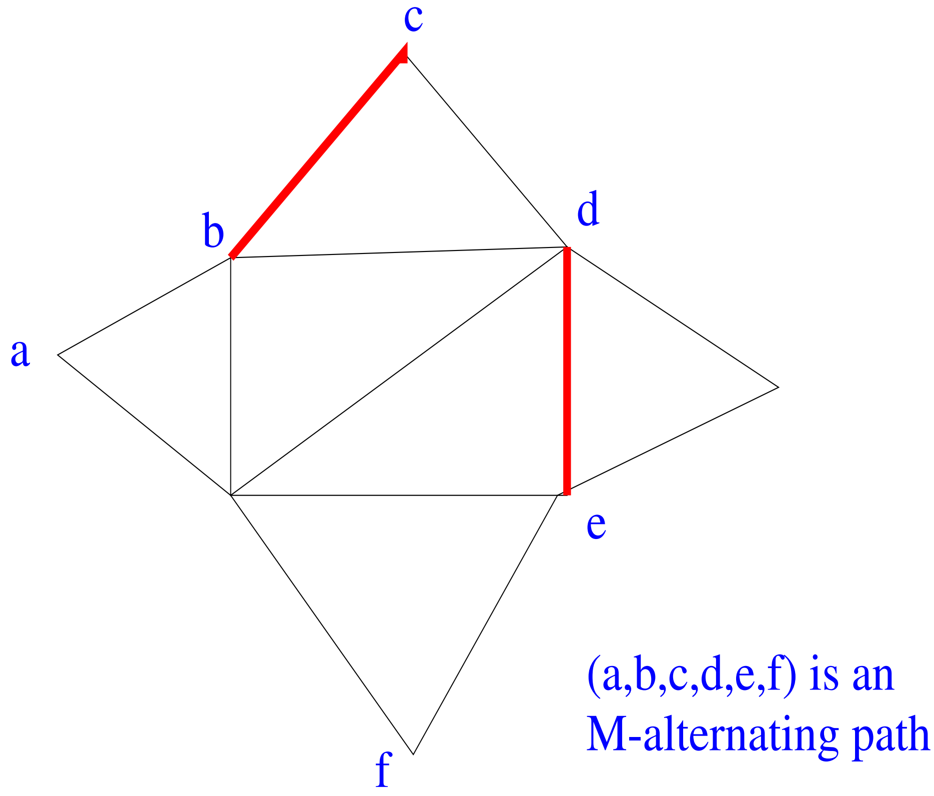
# Matchings

A *matching* $M$ of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





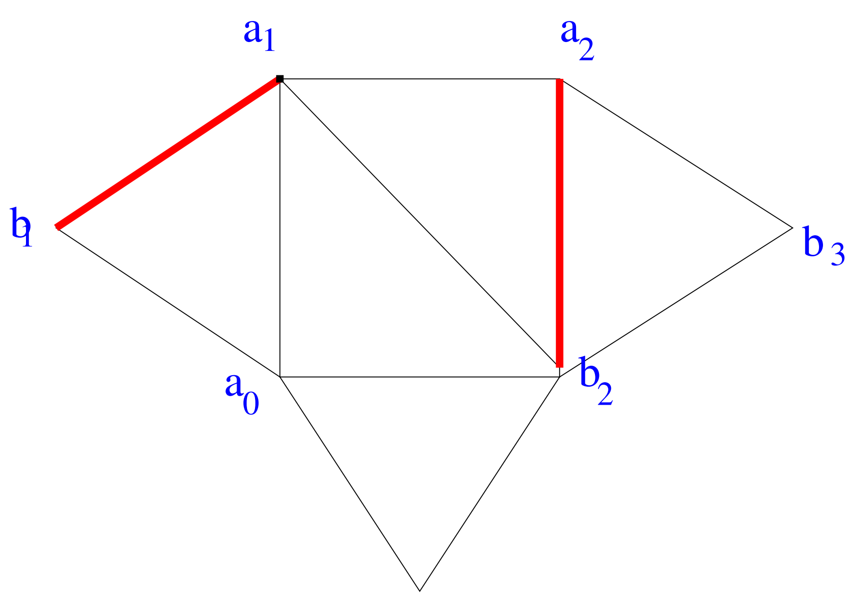




An $M$-alternating path joining 2 $M$-unsaturated vertices is called an $M$-augmenting path.

$M$ is a *maximum* matching of $G$ if no matching $M'$ has more edges.

**Theorem 1** *$M$ is a maximum matching iff $M$ admits no $M$-augmenting paths.*

**Proof** Suppose $M$ has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and $f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.




$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$

- $|M'| = |M| + 1$.
- $M'$ is a matching

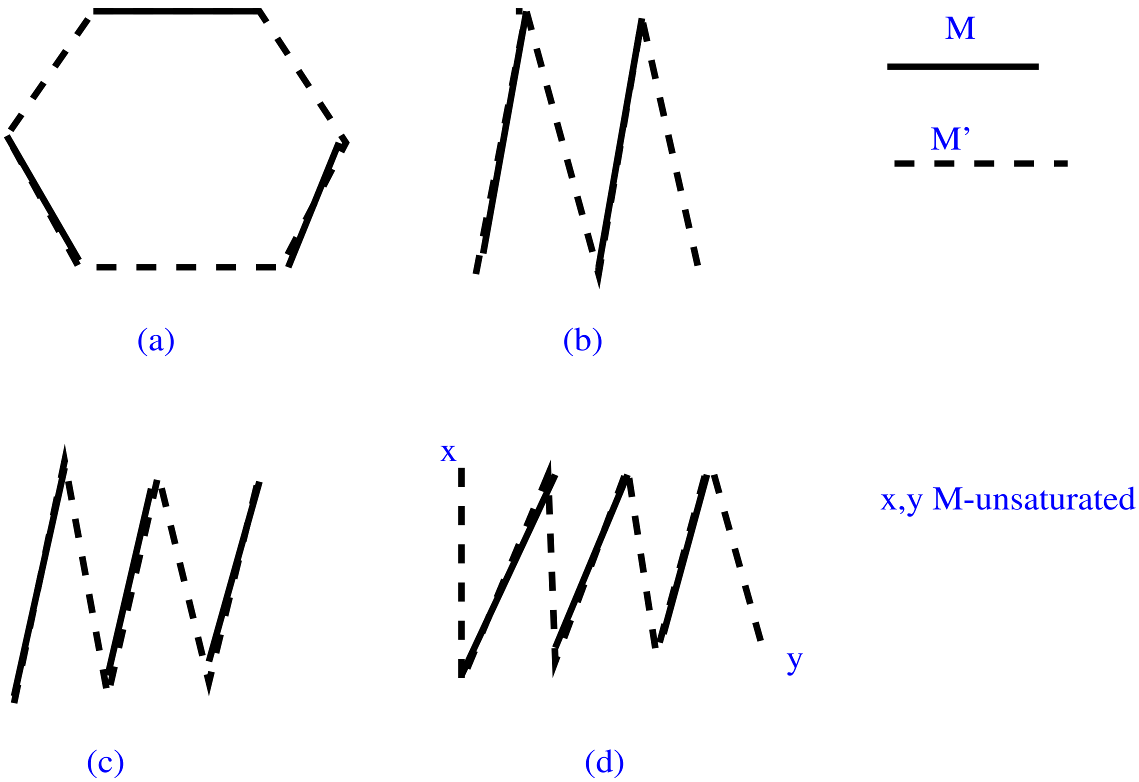
For $x \in V$ let $d_M(x)$ denote the degree of $x$ in matching $M$, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if $M$ has an augmenting path it is not maximum.

Suppose $M$ is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of $M, M'$.

Maximum degree of $H$ is 2 – $\leq \mathbf{1}$ edge from $M$ or $M'$. So $H$ is a collection of vertex disjoint alternating paths and cycles.



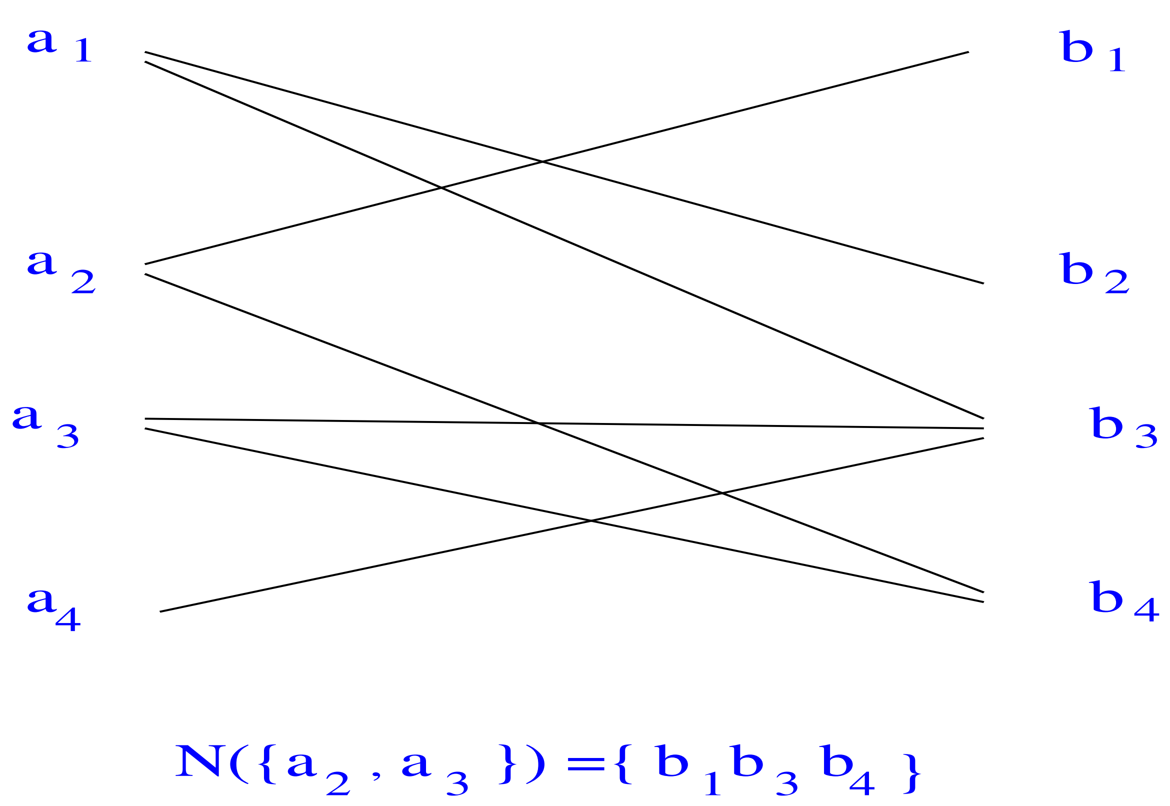

$|M'| > |M|$ impplies that there is at least one path of type (d).

Such a path is $M$-augmenting □

## Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition $A, B$.

For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.

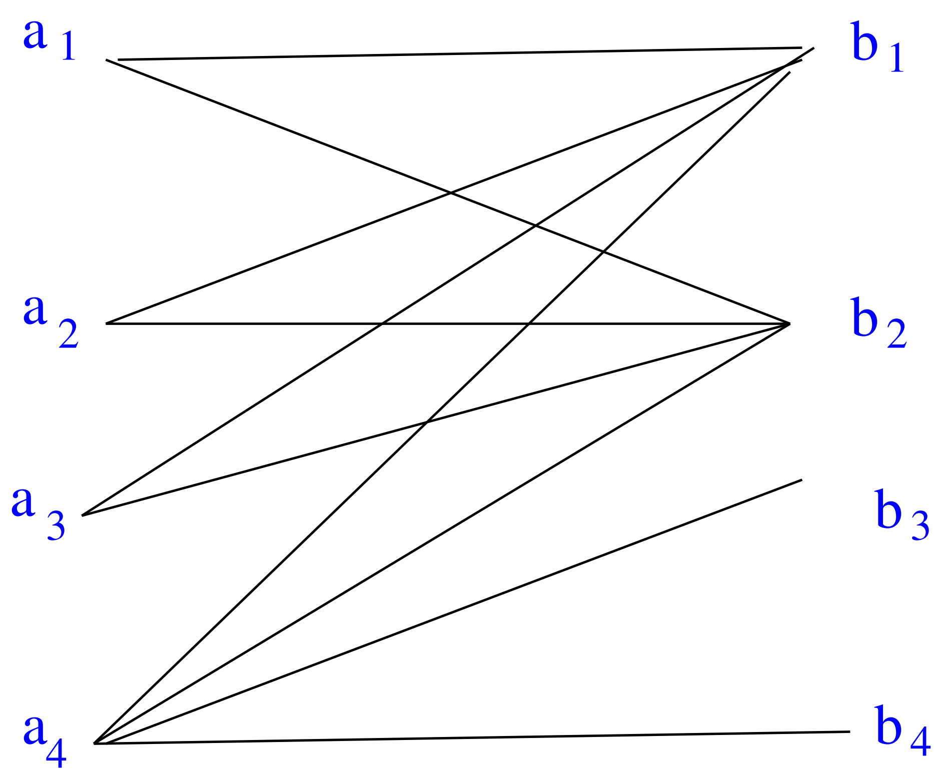



Clearly, $|M| \leq |A|, |B|$ for any matching $M$ of $G$.

## Hall's Theorem

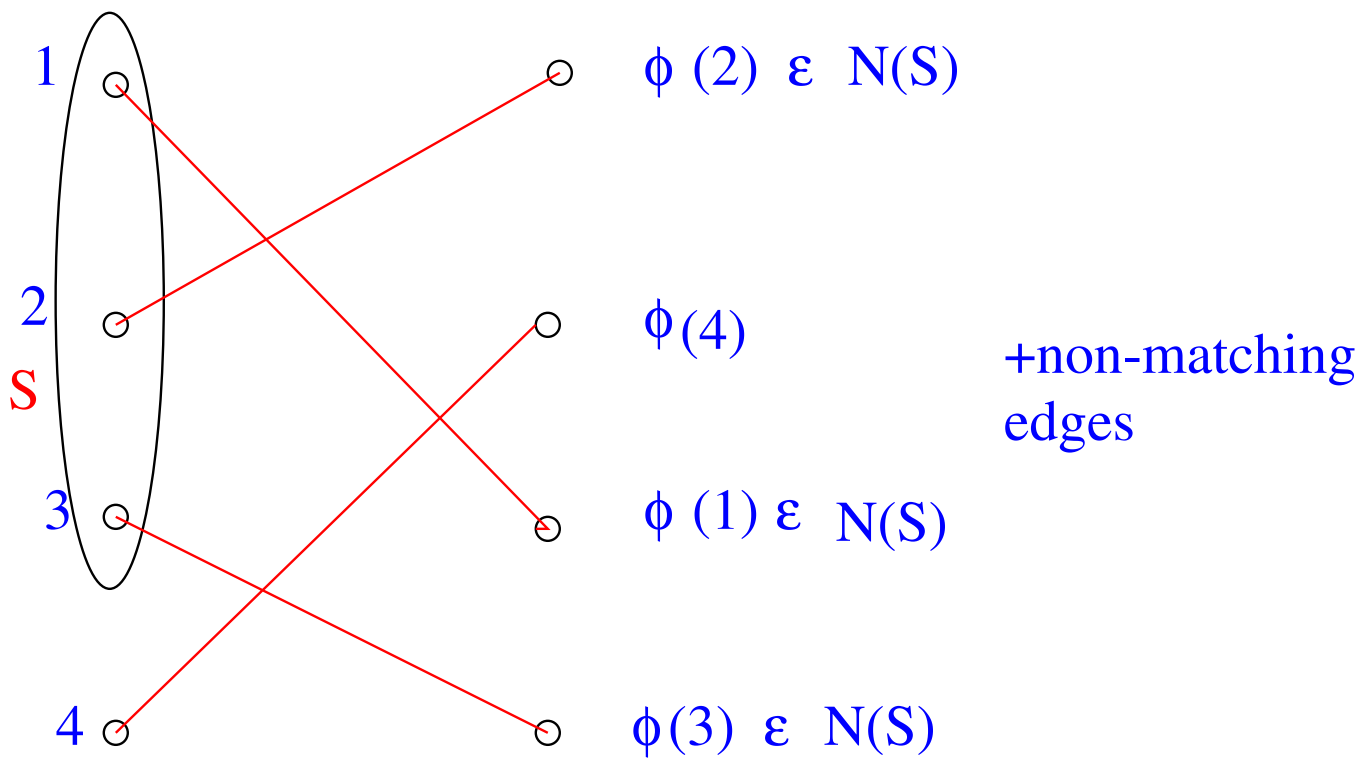
**Theorem 2** *$G$ contains a matching of size $|A|$ iff*

$$|N(S)| \geq |S| \qquad \forall S \subseteq A. \qquad (1)$$




$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of $a_1, a_2, a_3$ can be saturated by a matching.

**Only if**: Suppose $M = \{(a, \phi(a)) : \ a \in A\}$ saturates $A$.




$$
\begin{aligned}
|N(S)| &\geq |\{\phi(s) : \ s \in S\}| \\
&= |S|
\end{aligned}
$$

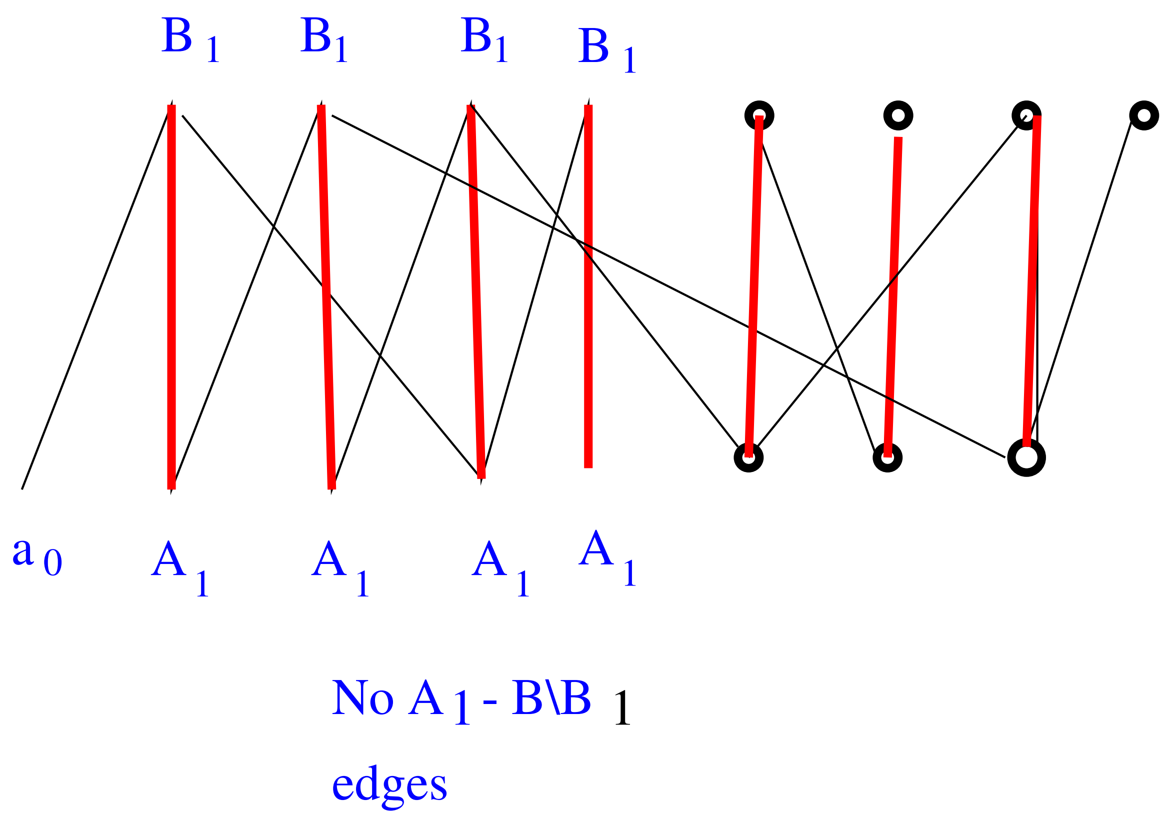
and so (1) holds.

**If**: Let $M = \{(a, \phi(a)) : \ a \in A'\}$ ($A' \subseteq A$) is a maximum matching. Suppose $a_0 \in A$ is $M$-unsaturated. We show that (1) fails.

Let

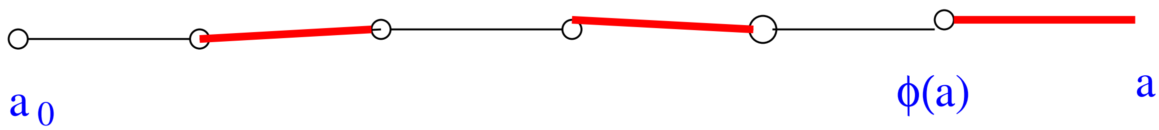
$A_1 = \{a \in A$ : such that $a$ is reachable from $a_0$ by an $M$-alternating path.$\}$

$B_1 = \{b \in B$ : such that $b$ is reachable from $a_0$ by an $M$-alternating path.$\}$



• $B_1$ is $M$-saturated else there exists an $M$-augmenting path.

• If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.

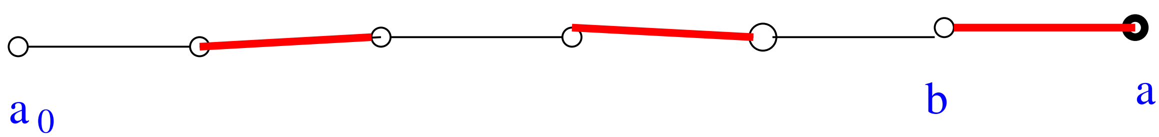



• If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.

So

$$|B_1| = |A_1| - 1.$$

• $N(A_1) \subseteq B_1$




So

$$|N(A_1)| = |A_1| - 1$$

and (1) fails to hold.

## Marriage Theorem

**Theorem 3** *Suppose* $G = (A \cup B, E)$ *is* $k$*-regular. (*$k \geq 1$*) i.e.* $d_G(v) = k$ *for all* $v \in A \cup B$*. Then* $G$ *has a perfect matching.*

**Proof**

$$k|A| = |E| = k|B|$$

and so $|A| = |B|$.

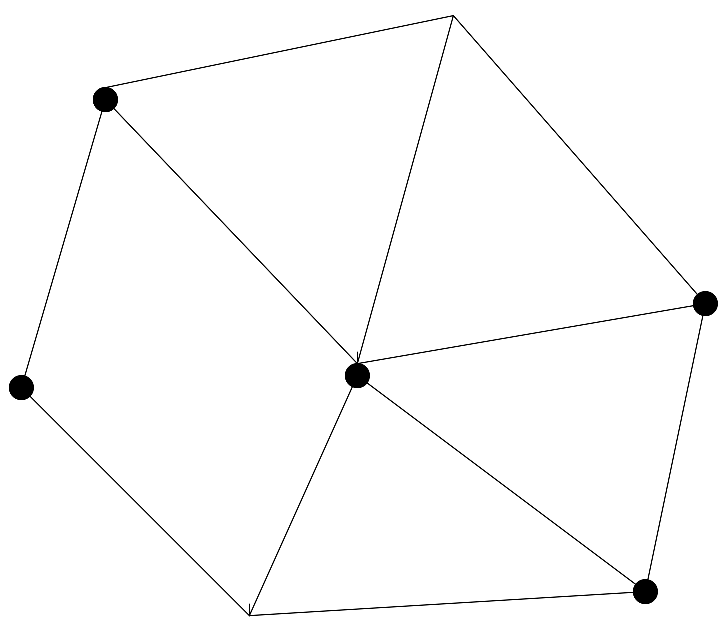
Suppose $S \subseteq A$. Let $m$ be the number of edges incident with $S$. Then

$$k|S| = m \leq k|N(S)|.$$

So (1) holds and there is a matching of size $|A|$ i.e. a perfect matching.

## Edge Covers

A set of vertices $X \subseteq V$ is a *covering* of $G = (V, E)$ if every edge of $E$ contains at least one endpoint in $X$.



$\{\bullet\}$ is a covering

**Lemma 1** *If $X$ is a covering and $M$ is a matching then $|X| \geq |M|$.*

**Proof** Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. $\square$

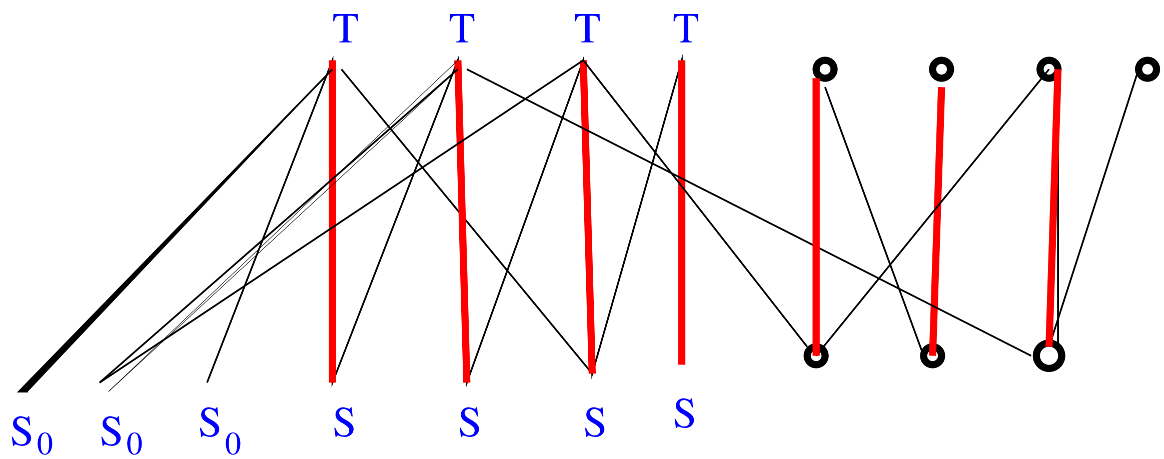
## Konig's Theorem

Let $\mu(G)$ be the maximum size of a matching.
Let $\beta(G)$ be the minimum size of a covering.
Then

$$\mu(G) \leq \beta(G).$$

**Theorem 4** *If $G$ is* bipartite *then $\mu(G) = \beta(G)$.*

**Proof** Let $M$ be a maximum matching.
Let $S_0$ be the $M$-unsaturated vertices of $A$.
Let $S \supseteq S_0$ be the $A$-vertices which are reachable from $S$ by $M$-alternating paths.
Let $T$ be the $M$-neighbours of $S \setminus S_0$.




Let $X = (A \setminus S) \cup T$.

• $|X| = |M|$.

$|T| = |S \setminus S_0|$. The remaining edges of $M$ cover $A \setminus S$ exactly once.

• $X$ is a cover.

There are no edges $(x, y)$ where $x \in S$ and $y \in B \setminus T$. Otherwise, since $y$ is $M$-saturated (no $M$-augmenting paths) the $M$-neighbour of $y$ would have to be in $S$, contradicting $y \notin T$. □