Inclusion-Exclusion

2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

So if $A_1, A_2 \subseteq A$ and $\bar{A}_i = A \setminus A_i$, $i = 1, 2$ then

$$|\bar{A}_1 \cap \bar{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3 sets:

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= |A| - |A_1| - |A_2| - |A_3| \\ &\quad + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

General Case

$A_1, A_2, \ldots, A_N \subseteq A$.

For $S \subseteq [N]$, $A_S = \bigcap_{i \in S} A_i$.

E.g. $A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18}$.

$A_\emptyset = A$.

Inclusion-Exclusion Formula:

$$\left| \bigcap_{i=1}^{N} \bar{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.$$

Simple example. How many integers in [1000] are not divisible by 5,6 or 8 i.e. what is the size of $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ below?

$$\begin{array}{rcll}
A = A_\emptyset & = & \{1, 2, 3, \ldots, \} & |A| = 1000 \\
A_1 & = & \{5, 10, 15, \ldots, \} & |A_1| = 200 \\
A_2 & = & \{6, 12, 18, \ldots, \} & |A_2| = 166 \\
A_3 & = & \{8, 16, 24, \ldots, \} & |A_2| = 125 \\
A_{\{1,2\}} & = & \{30, 60, 90, \ldots, \} & |A_{\{1,2\}}| = 33 \\
A_{\{1,3\}} & = & \{40, 80, 120, \ldots, \} & |A_{\{1,3\}}| = 25 \\
A_{\{2,3\}} & = & \{24, 48, 72, \ldots, \} & |A_{\{2,3\}}| = 41 \\
A_{\{1,2,3\}} & = & \{120, 240, 360, \ldots, \} & |A_{\{1,2,3\}}| = 8
\end{array}$$

$$\begin{aligned}
|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 1000 - (200 + 166 + 125) \\
&\qquad + (33 + 25 + 41) - 8 \\
&= 600.
\end{aligned}$$

Derangements

We must express the set of derangements D_n of $[n]$ as the intersection of the complements of sets.

We let $A_i = \{\text{permutations } \pi : \ \pi(i) = i\}$ and then

$$D_n = \left| \bigcap_{i=1}^{n} \bar{A}_i \right| .$$

We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n-1)!$: after fixing $\pi(1) = 1$ there are $(n-1)!$ ways of permuting $2, 3, \ldots, n$.

$|A_{\{1,2\}}| = (n-2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n-2)!$ ways of permuting $3, 4, \ldots, n$.

In general

$$|A_S| = (n - |S|)!$$

$$
\begin{aligned}
|D_n| &= \sum_{S\subseteq[n]} (-1)^{|S|}(n-|S|)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.
\end{aligned}
$$

Proof of inclusion-exclusion formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

Then

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^{N} \bar{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned}
\left| \bigcap_{i=1}^{N} \bar{A}_i \right| &= \sum_{x \in A} (1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) \\
&= \sum_{x \in A} \sum_{S \subseteq [N]} (-1)^{|S|} \prod_{i \in S} \theta_{x,i} \\
&= \sum_{S \subseteq [N]} (-1)^{|S|} \sum_{x \in A} \prod_{i \in S} \theta_{x,i} \\
&= \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|.
\end{aligned}$$

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n, other than 1.

$\phi(12) = 4$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n.

$$A_i = \{x \in [n] : \ p_i \text{ divides } x\}, \qquad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \bar{A}_i \right|$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i} \qquad S \subseteq [k].$$

$$\begin{aligned} \phi(n) &= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} \\ &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \end{aligned}$$

Surjections

Fix n, m. Let

$$A = \{f : [n] \to [m]\}$$

Thus $|A| = m^n$. Let

$$F(n,m) = \{f \in A : \ f \text{ is onto } [m]\}.$$

How big is $F(n,m)$? – note that

$$S(n,k) = |F(n,k)|/k!.$$

Let

$$A_i = \{f \in F : \ f(x) \neq i, \forall x \in [n]\}.$$

Then

$$F(n,m) = \bigcap_{i=1}^{m} \bar{A}_i.$$

For $S \subseteq [m]$

$$\begin{aligned} A_S &= \{f \in A : \ f(x) \notin S, \forall x \in [n]\}. \\ &= \{f : [n] \to [m] \setminus S\}. \end{aligned}$$

So

$$|A_S| = (m - |S|)^n.$$

Hence

$$\begin{aligned} F(n,m) &= \sum_{S \subseteq [m]} (-1)^{|S|}(m - |S|)^n \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n. \end{aligned}$$