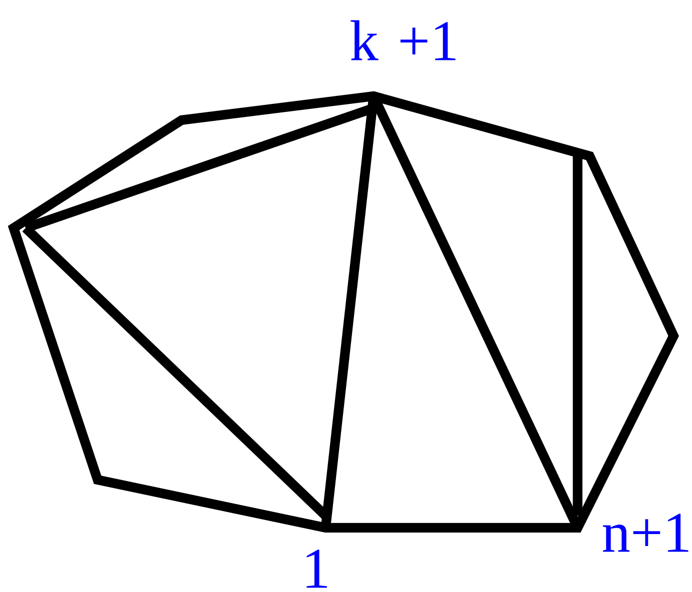
Triangulation of n-gon

Let

$$\begin{aligned} a_n &= \text{no. of triangulations of } P_{n+1} \\ &= \sum_{k=0}^{n} a_k a_{n-k} \qquad n \geq 2 \end{aligned} \tag{1}$$

$a_0 = 0$, $a_1 = a_2 = 1$.

Solution – use generating functions.



Explanation of (1):

$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.

There are a_k ways of triangulating $1, 2, \ldots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \ldots, n+1, k+1$.

If $a_0, a_1, \ldots, a_n$ is a sequence of real numbers then its **(ordinary) generating function** $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

and we write

$$a_n = [x^n]a(x).$$

$a_n = 1$

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$a_n = n + 1$.

$$a(x) = \frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots+(n+1)x^n+\cdots$$

$a_n = n$.

$$a(x) = \frac{x}{(1-x)^2} = x+2x^2+3x^3+\cdots+nx^n+\cdots$$

$a_n = \binom{\alpha}{n}$

$$a(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

$a_n = \binom{c+n-1}{n}$

$$a(x) = \frac{1}{(1-x)^c} = \sum_{n=0}^{\infty} \binom{-c}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{c+n-1}{n} x^n.$$

Simple properties of generating functions

- $a_n = [x^n]a(x)$ and $b_n = [x^n]b(x)$.
  $a(x) + b(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$ and so
  $a_n + b_n = [x^n]a(x) + b(x)$.

- $a_n = [x^n]a(x)$. Then

$$xa(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

  So $a_{n-1} = [x^n]xa(x)$.
  e.g. $n + 1 = [x^n](1 - x)^{-2}$ and
  $n = [x^n]x(1 - x)^{-2}$.

- $a_n = [x^n]a(x)$. Then

$$a'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

  and so $(n + 1)a_{n+1} = [x^n]a'(x)$.
  e.g. $[n + 1] = [x^n](1 - x)^{-2}$ and so
  $(n + 1)(n + 2) = [x^n]2(1 - x)^{-3}$.

Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad n \geq 2.$$

$a_0 = 1, a_1 = 9.$

$$\sum_{n=2}^{\infty} (a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \qquad (2)$$

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= a(x) - a_0 - a_1 x \\
&= a(x) - 1 - 9x. \\
\sum_{n=2}^{\infty} 6a_{n-1} x^n &= 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 6x(a(x) - a_0) \\
&= 6x(a(x) - 1). \\
\sum_{n=2}^{\infty} 9a_{n-2} x^n &= 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 9x^2 a(x).
\end{aligned}$$

So (2) gives

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

So

$$\begin{aligned} a(x) &= \frac{1 + 3x}{1 - 6x + 9x^2} \\ &= \frac{1 + 3x}{(1 - 3x)^2} \\ &= \sum_{n=0}^{\infty} (n+1)3^n x^n + 3x \sum_{n=0}^{\infty} (n+1)3^n x^n \\ &= \sum_{n=0}^{\infty} (n+1)3^n x^n + \sum_{n=0}^{\infty} n 3^n x^n \\ &= \sum_{n=0}^{\infty} (2n+1)3^n x^n. \end{aligned}$$

So

$$a_n = (2n+1)3^n.$$

$$a_n - 3a_{n-1} = n^2 \qquad n \geq 1.$$

$$a_0 = 1.$$

$$\begin{aligned}
\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n &= \sum_{n=1}^{\infty} n^2 x^n \\
\sum_{n=1}^{\infty} n^2 x^n &= \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} n x^n \\
&= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= \frac{x + x^2}{(1-x)^3} \\
\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n &= a(x) - 1 - 3xa(x) \\
&= a(x)(1-3x) - 1.
\end{aligned}$$

$$
\begin{aligned}
a(x) &= \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x} \\
&= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D+1}{1-3x}
\end{aligned}
$$

where

$$
\begin{aligned}
x+x^2 \;\cong\; & A(1-x)^2(1-3x) + B(1-x)(1-3x) \\
& + C(1-3x) + D(1-x)^3.
\end{aligned}
$$

Then

$$A = -1/2,\ B = 0,\ C = -1,\ D = 3/2.$$

So

$$
\begin{aligned}
a(x) &= \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x} \\
&= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}\binom{n+2}{2}x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n
\end{aligned}
$$

So

$$
\begin{aligned}
a_n &= -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2}3^n \\
&= -\frac{3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2}3^n.
\end{aligned}
$$

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n,\ b(x)) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned} a(x)b(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots) \times \\ &\qquad (b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \\ &\qquad (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

$a(x) = b(x) = 1/(1-x)$. $a_n = b_n = 1$, $n \geq 0$.

$$c_n = \sum_{k=0}^{n} 1 \times 1 = n + 1.$$

So

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

$$\begin{aligned} a(x) &= e^{\alpha x},\ b(x) = \frac{1}{1-x}. \\ c_n &= \sum_{k=0}^{n} \frac{\alpha^k}{k!}. \\ \frac{e^{\alpha x}}{1-x} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\alpha^k}{k!} \right) x^n. \end{aligned}$$

Solution of polygon triangulation problem.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n.$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n \\ &= a(x)^2. \end{aligned}$$

So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \text{ or } \frac{1-\sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}}\binom{2n-2}{n-1}(-4x)^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Derangements

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n-k$ elements.

So

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \qquad (3) \end{aligned}$$

Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m.$$

From (3) we have

$$\begin{aligned}
\frac{1}{1-x} &= e^x d(x) \\
d(x) &= \frac{e^{-x}}{1-x} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right) x^n.
\end{aligned}$$

So

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$