# Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$is an infinite sequence.

A recurrence recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \tag{1}$$

The whole sequence is determined by (1) and the values of $a_0, a_1, \ldots, a_{k-1}$.

## Linear Recurrence

(1) Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2.$$

$a_0 = a_1 = 1.$

(2)

$$b_n = |B_n| = |\{x \in \{a,b,c\}^n : \text{ } aa \text{ does not occur in } x\}|.$$

$b_1 = 3 : \ a \ b \ c$

$b_2 = 8 : \ ab \ ac \ ba \ bb \ bc \ ca \ cb \ cc$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

Let

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

where $B_n^{(\alpha)} = \{x \in B_n : \ x_1 = \alpha\}$ for $\alpha = a, b, c$.

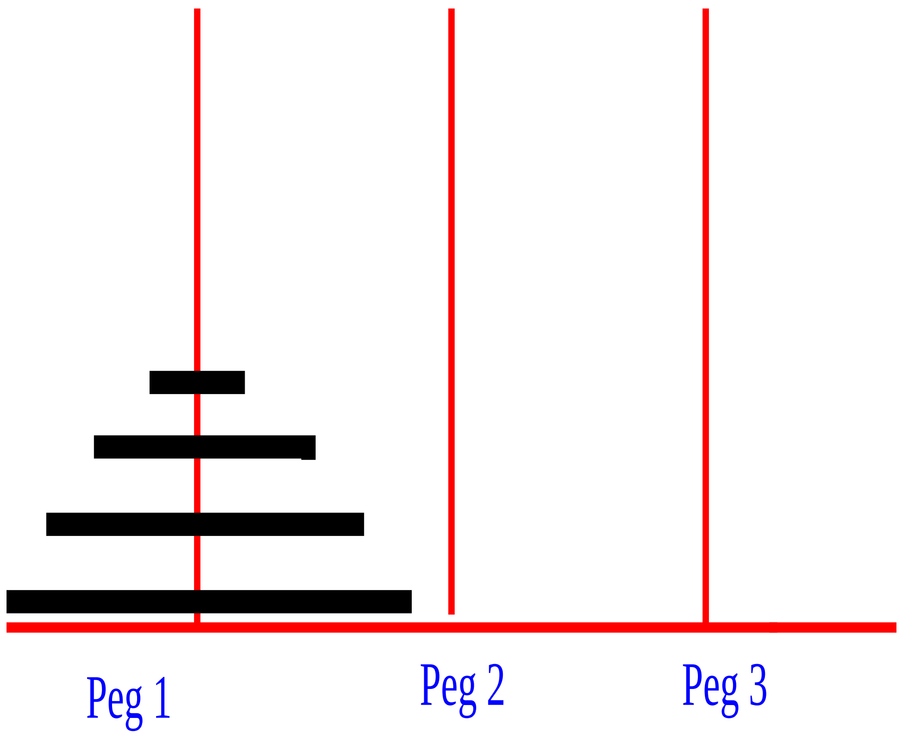
Now $|B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}|$. This is because the map $f : B_n^{(b)} \to B_{n-1}$ defined by

$$f(bx_2x_3 \ldots x_n) = x_2x_3 \ldots x_n$$

is a bijection.

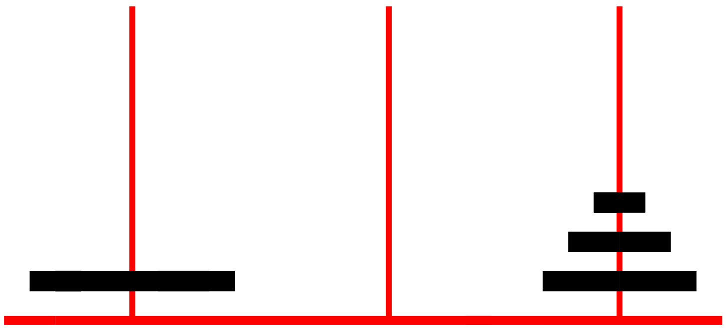
$B_n^{(a)} = \{x \in B_n : \ x_1 = a$ and $x_2 = b$ or $c\}$. Thus the map $g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(b)}$ defined by $g(ax_2x_3 \ldots x_n) = x_2x_3 \ldots x_n$ is a bijection. Hence, by the above, $|B_n^{(a)}| = 2|B_{n-2}|$.

## Towers of Hanoi

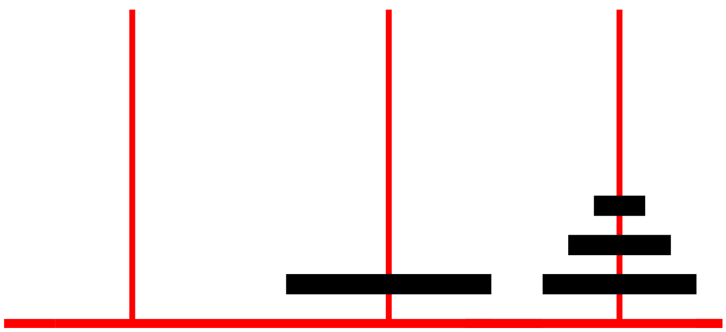



$H_n$ is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.

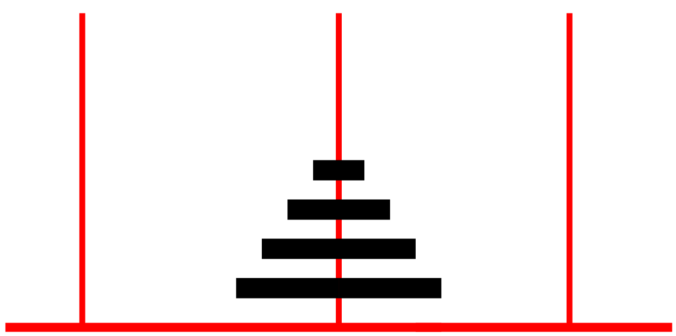
$$H_n = 2H_{n-1} + 1$$



$H_{n-1}$ moves



1 move



$H_{n-1}$ moves

$A$ has $n$ dollars. Everyday $A$ buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for $A$ to spend his money?

Ex. BBPIIPBI represents "Day 1, buy Bun. Day 2, buy Bun etc.".

$$\begin{aligned} u_n &= \text{number of ways} \\ &= u_{n,B} + u_{n,I} + u_{n,P} \end{aligned}$$

where $u_{n,B}$ is the number of ways where $A$ buys a Bun on day 1 etc.

$u_{n,B} = u_{n-1}, \; u_{n,I} = u_{n,P} = u_{n-2}$.

So

$$u_n = u_{n-1} + 2u_{n-2},$$

and

$$u_0 = u_1 = 1.$$

## Solution of Fibonacci Recurrence

We find solution to

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2. \tag{2}$$

$$a_0 = a_1 = 1. \tag{3}$$

First we "guess" solution $a_n = \xi^n$, $\xi \neq 0$ to (2).

$$\xi^n = \xi^{n-1} + \xi^{n-2}. \tag{4}$$

or

$$\xi^2 - \xi - 1 = 0$$

or

$$\xi = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}.$$

(4) is called the *characteristic equation* of the recurrence.

We observe that if the sequences $u_n$, $v_n$ both satisfy (2) then so does the sequence $c_1u_n + c_2v_n$ for any $c_1, c_2$.

Let $w_n = c_1u_n + c_2v_n$. Then

$$w_n - (w_{n-1} + w_{n-2}) =$$
$$c_1u_n + c_2v_n - ((c_1u_{n-1} + c_2v_{n-1}) + (c_1u_{n-2} + c_2v_{n-2})) =$$
$$c_1(u_n - (u_{n-1} + u_{n-2}) + c_2(v_n - (v_{n-1} + v_{n-2}) = 0.$$

Applying this with $u_n = ((1 + \sqrt{5})/2)^n$ and $v_n = ((1 - \sqrt{5})/2)^n$ we deduce that

$$c_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

satisfies (2) for any $c_1, c_2$.

We choose $c_1, c_2$ so that (3) also holds.

$$\begin{aligned} c_1 + c_2 &= 1 \qquad (n = 0) \\ c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} &= 1 \qquad (n = 1) \end{aligned}$$

So,

$$c_1 = \frac{1}{\sqrt{5}} \frac{1+\sqrt{5}}{2}, \; c_2 = -\frac{1}{\sqrt{5}} \frac{1-\sqrt{5}}{2}$$

and

$$a_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^{n+1} .$$

In general to solve

$$a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} = 0 \qquad n \geq 2$$

$$a_0 = \beta_0,\ a_1 = \beta_1,$$

we "guess" $a_n = \xi^n$ which gives us

$$\xi^n + \alpha_1 \xi^{n-1} + \alpha_2 \xi^{n-2} = 0$$

or

$$\xi^2 + \alpha_1 \xi + \alpha_2 = 0 \qquad (5)$$

Let $\xi_1, \xi_2$ be the roots of this equation. Put

$$a_n = c_1 \xi_1^n + c_2 \xi_2^n$$

where

$$\begin{aligned} c_1 + c_2 &= \beta_0 \qquad (n = 0) \qquad (6) \\ c_1\xi_1 + c_2\xi_2 &= \beta_1 \qquad (n = 1) \end{aligned}$$

Consider

$$u_n = u_{n-1} + 2u_{n-2} \qquad n \geq 2$$

$$u_0 = 1,\ u_1 = 1.$$

We solve

$$\xi^2 - \xi - 2 = 0$$

or

$$\xi = 2 \text{ or } -1.$$

We find $c_1, c_2$ such that

$$\begin{aligned} c_1 + c_2 &= 1 \qquad (n = 0) \\ 2c_1 - c_2 &= 1 \qquad (n = 1) \end{aligned}$$

$$c_1 = 2/3,\ c_2 = 1/3.$$

$$u_n = (2^{n+1} + (-1)^n)/3.$$

Consider

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 3$$

$$b_1 = 3,\ b_2 = 8.$$

We solve

$$\xi^2 - 2\xi - 2 = 0$$

or

$$\xi = 1 + \sqrt{3} \text{ or } 1 - \sqrt{3}.$$

We find $c_1, c_2$ such that

$$\begin{aligned} c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) &= 3 \qquad (n=1) \\ c_1(1+\sqrt{3})^2 + c_2(1-\sqrt{3})^2 &= 8 \qquad (n=2) \end{aligned}$$

$$c_1 = \frac{2+\sqrt{3}}{2\sqrt{3}},\ c_2 = \frac{5-3\sqrt{3}}{6-2\sqrt{3}}.$$

$$b_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{5-3\sqrt{3}}{6-2\sqrt{3}}(1-\sqrt{3})^n.$$

## Towers of Hanoi

$H_n = 2H_{n-1} + 1,\ H_1 = 1.$

$$
\begin{aligned}
\frac{H_n}{2^n} &= \frac{H_{n-1}}{2^{n-1}} + \frac{1}{2^n} \\
\frac{H_{n-1}}{2^{n-1}} &= \frac{H_{n-2}}{2^{n-2}} + \frac{1}{2^{n-1}} \\
&\vdots \\
\frac{H_2}{2^2} &= \frac{H_1}{2} + \frac{1}{2^2}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{H_n}{2^n} &= \frac{H_1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} \\
&= 1 - \frac{1}{2^n}
\end{aligned}
$$

$$H_n = 2^n - 1$$

## Alternative approach to general case

$$a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} = 0$$

implies

$$a_n - (\xi_1 + \xi_2) a_{n-1} + \xi_1 \xi_2 a_{n-2} = 0$$

[$\alpha_1 = -(\xi_1 + \xi_2)$, $\alpha_2 = \xi_1 \xi_2$]

So

$$(a_n - \xi_1 a_{n-1}) - \xi_2 (a_{n-1} - \xi_1 a_{n-2}) = 0.$$

Put $b_n = a_n - \xi_1 a_{n-1}$

$$b_n - \xi_2 b_{n-1} = 0$$

and so

$$b_n = c \xi_2^n$$

$$a_n - \xi_1 a_{n-1} = c\xi_2^n.$$

Assume $\xi_1 \neq 0$.

$$\begin{aligned}
\frac{a_n}{\xi_1^n} - \frac{a_{n-1}}{\xi_1^{n-1}} &= c\left(\frac{\xi_2}{\xi_1}\right)^n. \\
\frac{a_{n-1}}{\xi_1^{n-1}} - \frac{a_{n-2}}{\xi_1^{n-2}} &= c\left(\frac{\xi_2}{\xi_1}\right)^{n-1}. \\
&\vdots \\
\frac{a_1}{\xi_1} - a_0 &= c\frac{\xi_2}{\xi_1}.
\end{aligned}$$

Summing these equations we obtain (for $\xi_1 \neq \xi_2$)

$$\begin{aligned}
\frac{a_n}{\xi_1^n} &= c\left(\left(\frac{\xi_2}{\xi_1}\right)^n + \left(\frac{\xi_2}{\xi_1}\right)^{n-1} + \cdots + \frac{\xi_2}{\xi_1}\right) + a_0. \qquad (7) \\
&= c\frac{\xi_2}{\xi_1}\frac{(\xi_2/\xi_1)^n - 1}{(\xi_2/\xi_1) - 1} + a_0.
\end{aligned}$$

Multiplying through by $\xi_1^n$ justifies the formula

$$a_n = c_1\xi_1^n + c_2\xi_2^n \qquad \xi_1 \neq \xi_2.$$

When $\xi_1 = \xi_2$ we obtain (from (7))

$$a_n = \xi_1^n(cn + a_0)$$

as the general form.

## Complex roots

If the characteristic equation has a complex root, then both of the roots are complex and $\xi_2 = \overline{\xi_1}$, i.e. the two roots are different.

In this case we do the same as in the case when the characteristic equation has two different real roots. (We did not utilize the fact that the roots were real!)

This means that we will try to find a solution of the form $c_1\xi_1^n + c_2\xi_2^n$.

## Example for the complex case

$$a_n = a_{n-1} - 2a_{n-2}, \quad a_0 = 1,\ a_1 = 2.$$

We solve $\xi^2 - \xi + 2 = 0$ or

$$\xi_1 = \frac{1+\sqrt{7}i}{2},\ \xi_2 = \frac{1-\sqrt{7}i}{2}.$$

Since the roots are different, $a_n = c_1\xi_1^n + c_2\xi_2^n$.
We find $c_1$ and $c_2$ such that

$$c_1 + c_2 = 1,\ \text{and}$$

$$c_1\frac{1+\sqrt{7}i}{2} + c_2\frac{1-\sqrt{7}i}{2} = 2, \quad \text{i.e.,}$$

$$c_1 = \frac{7-3\sqrt{7}i}{14},\ \text{and}\ c_2 = \frac{7+3\sqrt{7}i}{14} \quad \text{i.e.,}$$

$$a_n = \frac{7-3\sqrt{7}i}{14}\left(\frac{1+\sqrt{7}i}{2}\right)^n + \frac{7+3\sqrt{7}i}{14}\left(\frac{1-\sqrt{7}i}{2}\right)^n.$$

Check that $a_n$ above is always an integer!