21-301 Combinatorics
Homework 9
Due: Wednesday, November 15

1. The width $w(P)$ of a finite poset is equal to the size of the maximum anti-chain. The hight $h(P)$ is the size of the largest chain. Prove that $w(P)h(P) \geq |P|$.

   Solution: Any cover of P by chains needs at least $|P|/h(P)$ chains. Dilworth's theorem then implies that $w(P) \geq |P|/h(P)$.

2. Several companies send members to a conference; the ith company sends m_i members. During the conference, several workshops are organized simultaneously; the jth workshop can accept at most n_j participants. The organizers want to assign participants to workshops so that no two members of the same company are in the same workshop. (The workshop need not be full.) Each member attends one workshop.

   a) Show how to use a flow network for testing if the constraints may be satisfied.
   b) If there are p companies and q workshops indexed in such a way that $m_1 \geq \cdots \geq m_p$ and $n_1 \leq \cdots \leq n_q$. Show that the constraints can be satisfied if $a(q-b) + \sum_{j=1}^{b} n_j \geq \sum_{i=1}^{a} m_i$ for all $0 \leq a \leq p$, $0 \leq b \leq q$.

   Solution: The network has vertices s, t, $C = \{c_1, c_2, \ldots, c_p\}, W = \{w_1, w_2, \ldots, w_q\}$. There is an edge (s, c_i) of capacity m_i for $i = 1, 2, \ldots, p$ and an edge (w_j, t) of capacity n_j for $j = 1, 2, \ldots, q$. In addition there is an edge (c_i, w_j) of capacity one for all i, j. The constraints can be satisfied iff there is a flow of value $\sum_{i=1}^{p} m_i$. Now consider the minimum size of a cut $(S : \bar{S})$. Let $a = |C \cap S|, b = |W \cap S|$. Then, the capacity of the cut is

$$\begin{aligned}
cap(S : \bar{S}) &= \sum_{i \in C \cap \bar{S}} m_i + \sum_{j \in W \cap S} n_j + |C \cap S| \cdot |W \cap \bar{S}| \\
&\geq \sum_{i=a+1}^{p} m_i + \sum_{j=1}^{b} n_j + a(q-b) \\
&= \sum_{i=1}^{p} m_i + \left(\sum_{j=1}^{b} n_j + a(q-b) - \sum_{i=1}^{a} m_i \right) \\
&\geq \sum_{i=1}^{p} m_i,
\end{aligned}$$

   if the given condition is satisfied.

3. There are n teams playing some sport. At some point in the season we have the following statistics: $w(i), i \in [n]$ denotes the number of games that team i has one so far. $p(i,j), i \neq j \in [n]$ denotes the number of times team i will play team j in the rest of the season. Set up the question: "can team n still end up with the most wins?" as a network flow problem.

   Solution: The network has vertices s, t, $L = \{\{i,j\} : i, j \in [n-1] \text{ and } p(i,j) > 0\}$ and $R = [n-1]$. For each $\{i,j\} \in L$ add an edge $(s, \{i,j\})$ of capacity $p(i,j)$. For each $\{i, j\} \in L$ add edges $(\{i, j\}, i)$ and $(\{i, j\}), j)$ of capacity ∞. For each $i \in R$ add an edge (i, t) of capacity $W - w(i)$ where $W = w(n) + \sum_{j<n} p(j, n)$ is the maximum number of games that team n can win.

We can assume that $W \geq w(i)$, else team n cannot end up with the most wins. Finally note that integral flows that saturate the edges leaving L are in 1-1 correspondence with outcomes of future matches. Also, the flow out of $i \in R$ into t is the number of wins $g(i)$ by team i in the associated schedule. Such a flow also represents an outcome where no team wins more than team n, since we would have $g(i) \leq W - w(i)$. So what we do is find a maximum flow in the network and see if it saturates the edges leaving s.