21-301 Combinatorics
Homework 8
Due: Monday, November 6

1. Let the sequence of integers $M_1, M_2, M_3, \ldots,$ be defined by the recurrence relation

$$M_1 = 3 \text{ and } M_k = kM_{k-1} - k + 2 \qquad (k > 1).$$

Show by induction on k that if the edges of K_{M_k} are coloured in k colours then there is a monochromatic triangle.
(Hint: let V_i denote the set of vertices v for which edge $\{1, v\}$ has color i.)

Solution: The result is true for $k = 1$ as only one colour is used. Assume it is true for $k' < k$. Let $M = M_k$ and suppose that K_M is k-edge coloured. Let $V_i = \{v : (1, v) \text{ has colour } i\}$. There exists i such that $|V_i| \geq M_{k-1}$ else

$$kM_{k-1} - k + 1 = \sum_{i=1}^{k} |V_i| \leq kM_{k-1} - k$$

contradiction.

But if $|V_i| \geq M_{k-1}$ then either V_i contains an edge (x, y) of colour i and $(1, x, y)$ is monochromatic or V_i only uses $k - 1$ colours and contains a monochromatic triangle by induction.

2. Let $m = s(p - 1) + 1$ and $n = s^m(q - 1) + 1$. Show that every s-coloring of the edges of $K_{m,n}$ using s colors contains a monochromatic copy of $K_{p,q}$.
(Hint: let X, Y be the two parts of the bipartition in $K_{m,n}$. Begin by showing that there must be q vertices $Q \subseteq Y$ such that $c(x, y) = c(x, y')$ for all $x \in X$ if $y, y' \in Q$.)

Solution: label each vertex $y_j \in Y$ by an s-ary vector whose ith coordinate is the color of the edge (x_i, y_j). Now we apply the Pigeonhole Principle twice. Since there are s^m such distinct vectors, there must be q vertices in Y with the same vector $\mathbf{x}$ as label. Since only s colors are available as entries for the m coordinates of $\mathbf{x}$, $\mathbf{x}$ must use some color at least p times. The p vertices of X in those positions and the q vertices of Y with label $\mathbf{x}$ induce a monochromatic copy of $K_{p,q}$.

3. Prove that for $n = 2m$ sufficiently large, every 2-coloring of the edges of $K_{n,n}$ contains a monochromatic copy of $K_{p,p}$.
(Hint: suppose that $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_n\}$ are the two parts of the bipartition in $K_{n,n}$. Consider the 2-coloring of K_m induced by the colors of the edges (x_i, y_{j+m}) for $i, j \leq m$.)

Solution: We color edge $\{i, j\}$ of K_m with the color of (x_i, y_{j+m}). If $m \geq R(2p, 2p)$, then the induced coloring of K_m will contain a monochromatic copy of K_{2p} on that subset $S, |S| = 2p$. Let $S = S_1 \cup S_2$ where $|S_1| = |S_2| = p$ and $\max S_1 < \min S_2$. It follows that the bipartite graph induced by $x_i, i \in S_1$ and $y_{j+m}, j \in S_2$ is monochromatic. (We use $|S| = 2p$ to avoid dealing with edges of the form $(i, i + m)$.)