21-301 Combinatorics
Homework 7
Due: Monday, October 30

1. Use the pigeon-hole principle to show that for every integer $k \geq 1$ and prime $p \neq 2, 5$ there exists a power of p that ends with $000 \cdots 0001$ (k 0's).
(Hint: consider the sequence $p^\ell \mod 10^{k+1}, \ell = 1, 2, \ldots,$.)
Solution: If we consider the infinite sequence $u_\ell = p^\ell \mod 10^{k+1}$ for $\ell = 1, 2, \ldots,$ then by the PHP there exist $m < n$ such that $u_m = u_n$, In which case,

$$p^n - p^m = 10^{k+1}s \text{ or } p^{n-m}(p^m - 1) = 10^{k+1}s$$

for some positive integer s.

Now p and 10 are co-prime and therefore $p^m - 1 = 10^{k+1}s'$ for some positive integer s', and this implies the result.

2. Suppose that we two-color the edges of K_6 Red and Blue. Show that there are at least two monochromatic triangles.
Solution: Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining x to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.

3. Show that $r(C_4, C_4) = 6$ where C_4 denotes a cycle of length four.
Solution: (a) Color the edges of the 5-cycle (1,2,3,4,5,1) Red and the edges of the remaining 5-cycle (1,3,5,2,4,1) Blue. There are no mono-chromatic 4-cycles.

(b) From class, there is a monochromatic triangle; Assume {1, 3, 5} is a red triangle (odds are chosen on purpose here for clarity). Suppose some even vertex contains two odd red neighbors; say, {2, 1} and {2, 3} are red, then {1, 2, 3, 5, 1} is red. Next, suppose instead that two even vertices have two common odd blue neighbors, then the four edges involved form a blue cycle.

The only way to avoid one of the previous two scenarios is to have, between evens and odds, only three disjoint red edges and the rest blue; Assume then that {1, 4},{2, 5}, and {3, 6} are red and let {1, 2, 3, 4, 5, 6, 1} be a blue 6-cycle. There are now only three edges to consider (evens to evens); any of these would complete a red 4-cycle if colored red (for example: making {2, 4} red gives {1, 4, 2, 5, 1}), so color all even-to-even edges blue. Now {2, 3, 4, 6, 2} is a blue cycle, so we are done.