21-301 Combinatorics
Homework 5
Due: Wednesday, October 13

1. Subsets $A_i, B_i \subseteq [n]$, $i = 1, 2, \ldots, m$ satisfy (i) $A_i \cap B_i = \emptyset$ for all i and (ii) $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Show that
$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$
Solution: Let π be a random permutation of $[n]$ and for disjoint sets A, B define the event $\mathcal{E}(A, B)$ by
$$\mathcal{E}(A, B) = \{\pi : \ \max\{\pi(a) : \ a \in A\} < \min\{\pi(b) : \ b \in B\}\}.$$
The events $\mathcal{E}_i = \mathcal{E}(A_i, B_i)$, $i = 1, 2, \ldots, m$ are disjoint. Indeed, suppose that $\mathcal{E}(A_i, B_i)$ and $\mathcal{E}(A_j, B_j)$ occur. Let $x \in A_i \cap B_j$ and $y \in A_j \cap B_i$. x, y exist by (ii) and (i) implies that they are distinct. Then $\mathcal{E}(A_i, B_i)$ implies that $\pi(x) < \pi(y)$ and $\mathcal{E}(A_j, B_j)$ implies that $\pi(x) > \pi(y)$, contradiction.

Observe next that for two fixed disjoint sets A, B, $|A| = a, |B| = b$ there are exactly $\binom{n}{a+b} a!b!(n-a-b)!$ permutations that produce the event $\mathcal{E}(A, B)$. Indeed, there are $\binom{n}{a+b}$ places to position $A \cup B$. Then there are $a!b!$ that place A as the first a of these $a+b$ places. Finally, there are $(n-a-b)!$ ways of ordering the remaining elements not in $A \cup B$.

Thus
$$\begin{aligned}\Pr(\mathcal{E}(A_i, B_i)) &= \frac{n!}{(|A_i|+|B_i|)!(n-|A_i|-|B_i|)!}|A_i|!|B_i|!(n-|A_i|-|B_i|)!\frac{1}{n!} \\ &= \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}.\end{aligned}$$
But then the disjointness of the collection of events $\mathcal{E}(A_i, B_i)$ implies that
$$\sum_{i=1}^{m} \Pr(\mathcal{E}(A_i, B_i)) \leq 1.$$

2. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^{n} \varepsilon_i x_i$, ($\varepsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.
(Hint: use Sperner's lemma.)
Solution: For $A \subseteq [n]$ let $x_A = \sum_{i \in A} x_i - \sum_{i \notin A} x_i$. Let $\mathcal{A} = \{A : x_A \in J\}$. It is enough to show that $\mathcal{A}$ is a Sperner family. Indeed, if $A, B \in \mathcal{A}$ and $A \subset B$ then $x_B - x_A = 2\sum_{i \in B \setminus A} x_i \geq 2$. Thus we cannot have both $x_A, x_B \in J$.

3. We say that a family $\mathcal{A} \subseteq 2^{[n]}$ is p-intersecting if for $X, Y \in \mathcal{A}$ either $X \cap Y \neq \emptyset$ or there exist $x \in X, y \in Y$ such that $x, y \leq p$. Prove that if $\mathcal{A} \subseteq \binom{[n]}{k}$ is p-intersecting

and $k \le n/2$ then $|\mathcal{A}| \le \binom{n}{k} - \binom{n-p}{k}$.
(Hint: partition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1 = \{A \in \mathcal{A} : p \notin A\}$ and use induction on p.)

Solution: If $p = 1$ then the bound given is equal to $\binom{n-1}{k-1}$, from Pascal's triangle. Also, 1-intersecting implies intersecting and so we have a basis for our induction. if $p > 1$ then the induction hypothesis implies that $|\mathcal{A}_1| \le \binom{n-1}{k} - \binom{(n-1)-(p-1)}{k}$. Now $|\mathcal{A}_2| \le \binom{n-1}{k-1}$ and so the result follows from Pascal's triangle.