21-301 Combinatorics
Homework 6
Due: Monday, October 9

1. Subsets $A_i, B_i \subseteq [n]$, $i = 1, 2, \ldots, m$ satisfy (i) $A_i \cap B_i = \emptyset$ for all i and (ii) $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Show that
$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$
(Hint: Let π be a random permutation of $[n]$ and for disjoint sets A, B define the event $\mathcal{E}(A, B)$ by
$$\mathcal{E}(A, B) = \{\pi : \max\{\pi(a) : a \in A\} < \min\{\pi(b) : b \in B\}\}.$$
Show that the events $\mathcal{E}_i = \mathcal{E}(A_i, B_i)$, $i = 1, 2, \ldots, m$ are disjoint.)

2. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^{n} \varepsilon_i x_i$, ($\varepsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.
(Hint: For $A \subseteq [n]$ let $x_A = \sum_{i \in A} x_i - \sum_{i \notin A} x_i$. Let $\mathcal{A} = \{A : x_A \in J\}$. Use Sperner's lemma.)

3. We say that a family $\mathcal{A} \subseteq 2^{[n]}$ is p-intersecting if for $X, Y \in \mathcal{A}$ either $X \cap Y \neq \emptyset$ or there exist $x \in X, y \in Y$ such that $x, y \leq p$. Prove that if $\mathcal{A} \subseteq \binom{[n]}{k}$ is p-intersecting and $k \leq n/2$ then $|\mathcal{A}| \leq \binom{n}{k} - \binom{n-p}{k}$.
(Hint: partition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1 = \{A \in \mathcal{A} : p \notin A\}$ and use induction on p.)