21-301 Combinatorics
Homework 5
Due: Monday, October 2

1. Fix $k \geq 1$. We say that a family of sets $A_1, A_2, \dots, A_m \subseteq [n]$ is k-intersection safe if there do not exist $i \neq j$ and $\ell_1, \ell_2, \dots, \ell_k$ such that $i, j \notin \{\ell_1, \ell_2, \dots, \ell_k\}$ and $A_i \cap A_j \subseteq \bigcup_{t=1}^{k} A_{\ell_t}$. Show that there exist k-intersection safe families of size c_k^n for some $c_k > 1$.

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every v and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Use the local lemma to prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)

3. Show that if $4 \cdot \frac{k^2(n-1)}{k-1} \cdot \frac{1}{2^{1-k}} < 1$ then one can 2-color the integers $1, 2, \dots, n$ such that there is no mono-colored arithmetic progression of length k.
(An arithmetic progression of length k is a set $\{a, a+d, \dots, a+(k-1)d\}$.)