21-301 Combinatorics
Homework 3
Due: Monday, September 25

1. Suppose that $A_1, A_2, \ldots, A_n \subseteq A$ and $|A_i| = k$ for $i = 1, 2, \ldots, n$ and that q is a positive integer. Show that if $nq\left(1 - \frac{1}{q}\right)^k < 1$ then the elements of A can be q-colored so that each A_i contains an element of each color.

   Solution: Randomly color the elements of A with q colors. Let $\mathcal{E}_{i,j}$ be the event that A_i is missing color j and let $\mathcal{E}_i = \bigcup_{j=1}^q \mathcal{E}_{i,j}$ and let $\mathcal{E} = \bigcup_{i=1}^n \mathcal{E}_i$. We need to show that $\mathbf{P}(\mathcal{E}) < 1$. Now

$$\begin{aligned} P(\mathcal{E}) &\leq \sum_{i=1}^n \mathbf{P}(\mathcal{E}_i) \\ &\leq \sum_{i=1}^n \sum_{j=1}^q \mathbf{P}(\mathcal{E}_{i,j}) \\ &= \sum_{i=1}^n \sum_{j=1}^q \left(1 - \frac{1}{q}\right)^k \\ &= nq\left(1 - \frac{1}{q}\right)^k \\ &< 1. \end{aligned}$$

2. Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Show that G contains a dominating set of size at most $n\frac{1+\log(\delta+1)}{\delta+1}$.
   (S is a dominating set if every $v \notin S$ has a neighbor in S.)
   (Hint: Choose $S_1 \subseteq V$ by placing v into S_1 with probability p. Let S_2 denote the vertices in $V \setminus S_1$ that are not adjacent to a vertex in S_1. Choose p carefully and use $1 - p \leq e^{-p}$.)

   Solution: Following the hint we have $\mathbf{E}(|S_1|) = np$ and

$$\mathbf{E}(|S_2|) = \sum_{v \in V}(1-p)^{d(v)+1} \leq n(1-p)^{\delta+1} \leq ne^{-(\delta+1)p}.$$

   So

$$\mathbf{E}(|S|) \leq f(p) = np + ne^{-(\delta+1)p}.$$

   Now

$$f'(p) = n - n(\delta+1)e^{-(\delta+1)p} = 0 \text{ when } p = \frac{\log(\delta+1)}{\delta+1}.$$

   (This is a minimum since $f''(p) > 0$ here.)
   Then we have

$$f\left(\frac{\log(\delta+1)}{\delta+1}\right) = n\frac{\log(\delta+1)}{\delta+1} + \frac{n}{\delta+1}.$$

   Now S is a dominating set and G must contain a dominating set of size at most $\mathbf{E}(|S|)$.

3. Prove that there is an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with pairwise distinct real entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

The following inequalities might be useful:

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k \text{ and } 1 + x \leq e^x \text{ and } n! \geq \left(\frac{n}{e}\right)^n.$$

Solution: Let π be a random permutation of the rows of A. For $S \subseteq [n]$ we let $\mathcal{E}_{i,S}$ be the event that elements of column i restricted to the rows in S form an increasing sequence. Let $\mathcal{E}_i = \bigcup_{S \in \binom{[n]}{s}} \mathcal{E}_{i,S}$ be the event that column i contains an increasing subsequence of length at least $s = c\sqrt{n}$ and let $\mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i$. We need to show that $\mathbf{P}(\mathcal{E}) < 1$. But,

$$\begin{aligned}
\mathbf{P}(\mathcal{E}) &\leq \sum_{i=1}^{n} \sum_{S \in \binom{[n]}{s}} \mathbf{P}(\mathcal{E}_{i,S}) \\
&= \sum_{i=1}^{n} \sum_{S \in \binom{[n]}{s}} \frac{1}{s!} \\
&= n \binom{n}{s} \frac{1}{s!} \\
&\leq n \left(\frac{ne}{s}\right)^s \left(\frac{e}{s}\right)^s \\
&= n \left(\frac{e^2}{c^2}\right)^s \\
&= ne^{-2s} \\
&< 1,
\end{aligned}$$

for $c = e^2$ provided n is big enough so that $e^{cn^{1/2}} > n$. This is a good enough. I should have said for n large.

If one wants make the claim for all n then observe that $e^{e^2 n^{1/2}} > n$ for $n \geq 10$ say. Then all we need to do now is increase c so that $e^{cn^{1/2}} > n$ for $n \leq 10$ as well.