21-301 Combinatorics
Homework 3
Due: Monday, September 25

1. Suppose that $A_1, A_2, \ldots, A_n \subseteq A$ and $|A_i| = k$ for $i = 1, 2, \ldots, n$ and that q is a positive integer. Show that if $nq\left(1 - \frac{1}{q}\right)^k < 1$ then the elements of A can be q-colored so that each A_i contains an element of each color.

2. Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Show that G contains a dominating set of size at most $n\frac{1+\log(\delta+1)}{\delta+1}$.
   (S is a dominating set if every $v \notin S$ has a neighbor in S.)
   (Hint: Choose $S_1 \subseteq V$ by placing v into S_1 with probability p. Let S_2 denote the vertices in $V \setminus S_1$ that are not adjacent to a vertex in S_1. Choose p carefully and use $1 - p \leq e^{-p}$.)

3. Prove that there is an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with pairwise distinct real entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

The follwing inequalities might be useful:

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k \text{ and } 1 + x \leq e^x \text{ and } n! \geq \left(\frac{n}{e}\right)^n.$$