21-301 Combinatorics
Homework 3
Due: Monday, September 18

1. How many strings $a_1a_2\cdots a_n$ of length n consisting of 0's and 1's have no two consecutive 1's?

   Solution:
   (a) Let f_n be the number of strings made of zeros and ones with no two consecutive ones. If a_n ends in a 0, we have a_{n-1} possible strings. If a_n ends in a 1, it must end in a 01, so we have a_{n-2} possible strings. So,

   $$f_n = f_{n-1} + f_{n-2}.$$

   There is one empty valid sequence, two valid sequences of length 1 and three of length 2. Therefore $a_n = F_{n+2}$, where F_n is the n'th Fibonacci number.

   (b) How many strings $a_1a_2\cdots a_n$ of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?
   Solution:
   Define $b_1b_2\cdots b_{n-1}$ as follows: $b_i = 1$ iff $a_i = a_{i+1}$ and $b_i = 0$ otherwise. The string $b_1b_2\cdots b_{n-1}$ has no two consecutive ones. From (a) above, there are F_{n+1} strings of the defined type. For each string $b_1b_2\cdots b_{n-1}$ there are 2 strings $a_1a_2\cdots a_n$. So, the answer is $2F_{n+1}$.

2. Find a_n if

   $$a_n = 6a_{n-1} + 7a_{n-2}, a_0 = 2, a_1 = 10.$$

   Solution: Let $F(x) = \sum a_n x^n$. Then

   $$F(x) - 2 - 10x = 6x(F(x) - 2) + 7x^2F(x),$$

   $$F(x) = \frac{2-2x}{1-6x-7x^2},$$

   $$F(x) = \frac{3/2}{1-7x} + \frac{1/2}{1+x}.$$

   So

   $$a_n = \frac{3}{2}7^n + \frac{1}{2}(-1)^n.$$

3. A row of n lightbulbs, initially all off, must be turned on. Bulb 1 can be turned on or off at any time. For $i > 1$, bulb i can be turned on or off only when bulb $i-1$ is on and all earlier bulbs are off. Let a_n be the number of steps needed to turn all on; note that (a_n) begins (0,1,2,5,...). Let b_n be the number of steps to turn on bulb n for the first time. Let c_n be the number of steps until we have that bulb n is the only bulb on.

   (a) Show that $c_1 = 1$ and $c_n = 2c_{n-1} + 1$, for $n \geq 1$.
   (b) Why is $b_{n+1} = c_n + 1$?
   (c) Why is $a_n = b_n + a_{n-2}$ for $n \geq 2$?
   (d) Solve the recurrence for a_n.

[Hint: suppose that we represent the state of the n light bulbs by a $\{0,1\}$ vector $\mathbf{x}$ of length n where $x_i = 1$ iff light-bulb i is on. Suppose that $\mathbf{x}(k)$ is the state after k steps. Argue that $\mathbf{x}(c_n - t) = \mathbf{x}(t) + (0, 0, \ldots, 0, 1)$ for $t = 0, 1, \ldots, c_{n-1}$.]

Solution:
(a) We go from (i) $0^n 0$ to (ii) $0^{n-1}10$ to (iii) $0^{n-1}11$ to (iv) $0^n 1$. In the moves (iv) we reverse the moves in (i)-(ii). This implies that $c_n = 2^n - 1$.
(b) This is just the move (ii) to (iii) in (a). This implies that $b_{n+1} = 2^n$
(c) Once we have $0^{n-1}11$, we can focus on changing the first $n-2$ 0's by 1's.
(d) $a_n - a_{n-2} = 2^{n-1}$. So, for $m \geq 1$,

$$a_{2m} = 2^{2m-1} + 2^{2m-3} + \cdots + 2 = \frac{2(4^m - 1)}{3}$$
$$a_{2m+1} = 2^{2m} + 2^{2m-2} + \cdots + 2^4 + 5 = \frac{4^{m+1} - 1}{3}.$$