21-301 Combinatorics
Homework 3
Due: Monday, September 18

1. (a) How many strings $a_1a_2\cdots a_n$ of length n consisting of 0's and 1's have no two consecutive 1's?

   (b) How many strings $a_1a_2\cdots a_n$ of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?
   [Hint: define the string $b_1b_2\cdots b_{n-1}$ as follows: $b_i = 1$ iff $a_i = a_{i+1}$.]

2. Find a_n if

$$a_n = 6a_{n-1} + 7a_{n-2}, a_0 = 2, a_1 = 10.$$

3. A row of n lightbulbs, initially all off, must be turned on. Bulb 1 can be turned on or off at any time. For $i > 1$, bulb i can be turned on or off only when bulb $i-1$ is on and all earlier bulbs are off. Let a_n be the number of steps needed to turn all on; note that (a_n) begins (0,1,2,5,...). Let b_n be the number of steps to turn on bulb n for the first time. Let c_n be the number of steps until we have that bulb n is the only bulb on.

   (a) Show that $c_1 = 1$ and $c_n = 2c_{n-1} + 1$, for $n \geq 1$.
   (b) Why is $b_{n+1} = c_n + 1$?
   (c) Why is $a_n = b_n + a_{n-2}$ for $n \geq 2$?
   (d) Solve the recurrence for a_n.

   [Hint: suppose that we represent the state of the n light bulbs by a $\{0, 1\}$ vector $\mathbf{x}$ of length n where $x_i = 1$ iff light-bulb i is on. Suppose that $\mathbf{x}(k)$ is the state after k steps. Argue that $\mathbf{x}(c_n - t) = \mathbf{x}(t) + (0, 0, \ldots, 0, 1)$ for $t = 0, 1, \ldots, c_{n-1}$.]