21-301 Combinatorics
Homework 2
Due: Monday, September 11

1. Let M be a multiset consisting of two each of n types of letters. How many permutations of M are there in which no two consecutive letters are the same?

   Solution: Suppose that $M = \{x_1, x_1, x_2, x_2, \ldots, x_n, x_n\}$. Let A_i be the set of permutations $\ldots, x_i, x_i, \ldots$ i.e. those permutation for which the two copies of x_i appear consecutively. Then if $S \subseteq \{1, 2, \ldots, n\}$, $|S| = s$,

   $$|A_S| = \binom{2n-s}{s} \cdot s! \cdot \frac{(2n-2s)!}{2^{n-s}}. \tag{1}$$

   Finish by plugging the expression in the I-E formula to give the number of relevant permutations is

   $$\sum_{s=0}^{n} \binom{n}{s} \cdot (-1)^s \cdot \binom{2n-s}{s} \cdot s! \cdot \frac{(2n-2s)!}{2^{n-s}}.$$

   Explanation for (1)**:** The binomial coefficient $\binom{2n-s}{s}$is the number of choices of s 1's and $2n-s$ 0's such that each 1 is followed by at least one 0. The 1 indicates the place of the first of the pair. The second factor $s!$ is the number of ways of allocating the pairs in S to the positions in the permutation.The remaining factor is the number of permutations of a multi-set of $n-s$ objects, each appearing exactly twice.

2. How many permutations π of $[n]$ are there such that $\pi(i) = j$ implies that $\pi(j) \neq i$. Here $i \neq j$.

   Solution: If $e = \{i, j\} \in X$ then let A_e be the set of permutations in which $\pi(i) = j$ and $\pi(j) = i$. Then if $S \subseteq X$ contains pairs that intersect then $A_S = \emptyset$. If S consists of s disjoint pairs then

   $$|A_S| = (n-2s)!. \tag{2}$$

   Thus the total number is

   $$\sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \binom{n}{2s} \frac{(2s)!}{s!2^s} (n-2s)! = n! \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \frac{1}{s!2^s} \sim n! e^{-1/2}. \tag{3}$$

   Explanation for (2)**:** a permutation in A_S fixes exactly $2s$ values of the permutation.
   Explanation for (3)**:** choose $2s$ values in $\binom{n}{2s}$ ways and then pair them up in $\frac{(2s)!}{s!2^s}$ ways and then complete the permutation in $(n-2s)!$ ways.

3. How many ways are there of pairing n boys with n girls so that the ith tallest boy is not paired with the ith tallest girl, for all $i = 1, 2, \ldots, n$.

   Solution: This is equal to the number of derangements of $\{1, 2, \ldots, n\}$.