21-301 Combinatorics
Homework 8
Due: Wednesday, November 24

1. Let $\mathcal{M}$ be a matroid (without loops) on E and suppose that $r(E) = \ell$. Suppose that $\ell < k \leq |E|$ and $\mathcal{B}_k$ is the set of k-subsets of E that contain a base of $\mathcal{M}$. Show that $\mathcal{B}_k$ forms the set of bases of another matroid.

   Solution: we show that $\mathcal{B}_k$ satisfies the basis exchange property. So, suppose that $B_1, B_2 \in \mathcal{B}_k$ and that $e \in B_1$. There must be a base B of $\mathcal{M}$ that contains e and is contained in B_1. Choose any basis $B' \subseteq B_2$. There must exist $f \in B'$ such that $(B \cup \{f\}) \setminus \{e\}$ is a basis of $\mathcal{M}$. This implies that $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}_k$.

2. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\} \subseteq \binom{E}{k}$. Suppose also that $|H_i \cap H_j| \leq k-2$ for $i \neq j$. Show that $\mathcal{B}_\mathcal{H} = \binom{E}{k} \setminus \mathcal{H}$ forms the set of bases of a matroid.
   ($\binom{E}{k} = \{S \subseteq E : |S| = k\}$.)

   Solution: Suppose that $B_1, B_2 \in \mathcal{B}_\mathcal{H}$ and let $e \in B_1 \setminus B_2$. If $B_1 \setminus B_2 = \{f\}$ then $(B_1 \cup \{f\}) \setminus \{e\} = B_2 \in \mathcal{B}_\mathcal{H}$. So suppose that $f_1, f_2 \in B_1 \setminus B_2$. Let $A_i = (B_1 \cup \{f_i\}) \setminus \{e\}$ for $i = 1, 2$. Now $|A_1 \cap A_2| = k-1$ and so at least one of the $A_i \notin \mathcal{H}$.

3. Let $\mathcal{M}$ be a hereditary system and suppose the the rank function r is submodular. Prove that $\mathcal{M}$ satisfies the Weak Elimination Property for circuits.
   (WEP: if C_1, C_2 are circuits and $e \in C_1 \cap C_2$ then there exists a circuit $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.)

   Solution: we have $r(C_i) = |C_i| - 1, i = 1, 2$. Also $r(C_1 \cap C_2) = |C_1 \cap C_2|$ since every proper subset of a circuit is independent. If $(C_1 \cup C_2) \setminus \{e\}$ contains no circuit then it is independent and so $r((C_1 \cup C_2) \setminus \{e\}) = |C_1 \cup C_2| - 1$ and this implies that $r(C_1 \cup C_2) \geq |C_1 \cup C_2| - 1$. But then we have the contradiction

   $$|C_1| - 1 + |C_2| - 1 = r(C_1) + r(C_2) \geq r(C_1 \cup C_2) + r(C_1 \cap C_2) = |C_1 \cup C_2| - 1 + |C_1 \cap C_2|.$$