21-301 Combinatorics
Homework 8
Due: Wednesday, November 24

1. Let $\mathcal{M}$ be a matroid (without loops) on E and suppose that $r(E) = \ell$. Suppose that $\ell < k \leq |E|$ and $\mathcal{B}_k$ is the set of k-subsets of E that contain a base of $\mathcal{M}$. Show that $\mathcal{B}_k$ forms the set of bases of another matroid.

2. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\} \subseteq \binom{E}{k}$. Suppose also that $|H_i \cap H_j| \leq k - 2$ for $i \neq j$. Show that $\mathcal{B}_{\mathcal{H}} = \binom{E}{k} \setminus \mathcal{H}$ forms the set of bases of a matroid.
   ($\binom{E}{k} = \{S \subseteq E : |S| = k\}$.)

3. Let $\mathcal{M}$ be a hereditary system and suppose the the rank function r is submodular. Prove that $\mathcal{M}$ satisfies the Weak Elimination Property for circuits.
   (WEP: if C_1, C_2 are circuits and $e \in C_1 \cap C_2$ then there exists a circuit $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.)