21-301 Combinatorics
Homework 7
Due: Wednesday, November 3

1. How many ways are there to arrange 2 M's, 4 A's, 5 T's and 6 H's under the condition that any arrangement and its reversal are to be considered the same.

   **Solution:** The group $G$ consists of $\{e, a\}$ where $a$ is a reflection through the middle of the word. Now

$$|Fix(e)| = \frac{17!}{2!4!5!6!} = 85765680$$
$$|Fix(a)| = \frac{8!}{1!2!2!3!} = 1680$$

   A sequence is in $Fix(a)$ if it is a palindrome i.e. looks the same backwards as forwards. It must have middle letter T. Then we arrange 1 M, 2 A's, 2 T"s and 3 H's in any order and then complete the sequence uniquely to a palindrome.

   Thus by Burnside's theorem, the number of sequences is $\frac{85765680+1680}{2} = 42883680$.

2. Find the set of $P$-positions for the take-away games with subtraction sets

   (a) $S = \{1, 3, 7\}$.
   (b) $S = \{1, 4, 6\}$.

   Suppose now that there are two piles and the rules for each pile are as above. Now find the $P$ positions for the two pile game where in one pile $S$ is as in (a) and the other pile is as in (b).

   **Solution:**

   (a) The first few numbers are

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $g_1(j)$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

   It is apparent that $g_1(j) = j \mod 2$ and this follows by an easy induction: If $j$ is even then $j - x, x \in S$ is odd and if $j$ is odd then $j - x, x \in S$ is even.

   (b) The first few numbers are

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $g_2(j)$ | 0 | 1 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 2 |

   So, we see that the pattern 0 1 0 1 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

   (c) The $P$-positions are those $j$ for which $g_1(j) \oplus g_2(j) = 0$. Thus $P = \{j : j \mod 10 \leq 3\}$.

3. In a take-away game, the set $S$ of the possible numbers of chips to remove is finite. Show that the Sprague-Grundy numbers satisfy $g(n) \leq |S|$ where $n$ is the number of chips remaining.

**Solution:** Observe that for any finite set $A$, $mex(A) \leq |A|$ since $mex(A) > |A|$ implies that $A \subseteq \{0, 1, 2, \ldots, |A|\}$ which is obviously impossible. In the take-away game $g(n)$ is the mex of a set of size at most $|S|$ and the result follows.