21-301 Combinatorics
Homework 6
Due: Wednesday, October 27

1. Let $r_n = r(3,3,\ldots,3)$ be the minimum integer such that if we n-color the edges of the complete graph K_N there is a monochromatic triangle.

   (a) Show that $r_n \le n(r_{n-1}-1)+2$.

   (b) Using $r_2 = 6$, show that $r_n \le \lfloor n!e \rfloor + 1$.

   Solution: Let $N = n(r_{n-1}-1)+2$ and consider an n-coloring σ of the edges of K_N. Now consider the $N-1$ edges incident to vertex N. There must be a color, n say, that is used at least r_{n-1} times, Pigeon Hole Principle. Now let $V \subseteq [N-1]$ denote the set of vertices v for which the edge $\{v, N\}$ is colored n. Consider the coloring of the edges of V induced by σ. If one of these $\{v_1, v_2\}$ has color N then it makes a triangle v_1, v_2, N with 3 edges colored n. Otherwise the edges of V only use $n-1$ colors and since $|V| \ge r_{n-1}$ we see by induction that V contains a mono-chromatic triangle.

   Solution: Divide the inequality (a) by $n!$ and putting $s_n = r_n/n!$ we obtain

   $$s_n \le s_{n-1} - \frac{1}{(n-1)!} + \frac{2}{n!}. \tag{1}$$

   We write this as

   $$\begin{aligned} s_n - s_{n-1} &\le -\frac{1}{(n-1)!} + \frac{2}{n!} \\ s_{n-1} - s_{n-2} &\le -\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \\ &\vdots \\ s_3 - s_2 &\le -\frac{1}{2!} + \frac{2}{3!} \end{aligned}$$

   Summing gives

   $$s_n - s_2 \le -\frac{1}{2!} + \frac{1}{n!} + \sum_{k=3}^{n} \frac{1}{k!} \le \frac{1}{n!} + e - 2.$$

   Now $s_2 = 3$ and multiplying the above by $n!$ gives $r_n \le n!e + 1$. We round down, as r_n is an integer.

2. Show that $r(C_4, C_4) = 6$, where C_4 denotes a cycle of length 4.

   Solution: (a) Color the edges of the 5-cycle (1,2,3,4,5,1) Red and the edges of the remaining 5-cycle (1,3,5,2,4,1) Blue. There are no mono-chromatic 4-cycles.

   (b) Each vertex is incident with at least 3 edges of the same color. So, we can assume that 1,2,3 each have at least 3 red neighbors N_1, N_2, N_3. If $N_1, N_2 \subseteq \{3,4,5,6\}$ then $|N_1 \cap N_2| \ge 2$ and then there is a C_4 containing vertices 1,2.

   We can sssume then that 1,2,3 form a red triangle. If $4 \in N_1 \cap N_2$ then we have that 1,3,2,4,1 is a red C_4.

So we can assume that $|N_i \cap N_j| = 1$ for all i, j and that $N_1 = \{2,3,4\}, N_2 = \{1,3,5\}, N_3 = \{1,2,6\}$. If $\{4,5\}$ is red then 1,2,5,4,1 is a red C_4. So we can assume that 4,5,6, form a blue triangle.

If $\{1,5\}$ is red then 1,5,3,2,1 is a red C_4. So we can assume that $\{1,5\}, \{1,6\}, \{2,4\}, \{2,6\}, \{3,4\}, \{3,5\}$ are all blue.

But then 1,5,4,6,1 is a blue C_4.

3. Use Dilworth's theorem to show that if in a bipartite graph $G = (A, B, E)$ we have that $|N(S)| \geq |S| - t$ for all $S \subseteq A$, then there is a matching of size at least $|A| - t$.

   Solution: Let $G = (A \cup B, E)$ be a bipartite graph which satisfies the given condition. Define a poset $P = A \cup B$ and define $<$ by $a < b$ only if $a \in A, b \in B$ and $(a, b) \in E$. Suppose that the largest anti-chain in P is $A = \{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k\}$ and let $s = h + k$.

   Now
$$N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}$$
   for otherwise A will not be an anti-chain. From the given condition we see that
$$|B| - k \geq h - t \text{ or equivalently } |B| \geq s - t.$$

   Now by Dilworth's theorem, P is the union of s chains: A matching M of size m, $|A| - m$ members of A and $|B| - m$ members of B. But then
$$m + (|A| - m) + (|B| - m) = s \leq |B| + t$$
   and so $m \geq |A| - t$.