

21-301 Combinatorics

Homework 5

Due: Wednesday, October 13

1. Subsets $A_i, B_i \subseteq [n]$, $i = 1, 2, \dots, m$ satisfy (i) $A_i \cap B_i = \emptyset$ for all i and (ii) $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Show that

$$\sum_{i=1}^m \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Solution: Let π be a random permutation of $[n]$ and for disjoint sets A, B define the event $\mathcal{E}(A, B)$ by

$$\mathcal{E}(A, B) = \{\pi : \max\{\pi(a) : a \in A\} < \min\{\pi(b) : b \in B\}\}.$$

The events $\mathcal{E}_i = \mathcal{E}(A_i, B_i)$, $i = 1, 2, \dots, m$ are disjoint. Indeed, suppose that $\mathcal{E}(A_i, B_i)$ and $\mathcal{E}(A_j, B_j)$ occur. Let $x \in A_i \cap B_j$ and $y \in A_j \cap B_i$. x, y exist by (ii) and (i) implies that they are distinct. Then $\mathcal{E}(A_i, B_i)$ implies that $\pi(x) < \pi(y)$ and $\mathcal{E}(A_j, B_j)$ implies that $\pi(x) > \pi(y)$, contradiction.

Observe next that for two fixed disjoint sets A, B , $|A| = a, |B| = b$ there are exactly $\binom{n}{a+b} a! b! (n - a - b)!$ permutations that produce the event $\mathcal{E}(A, B)$. Indeed, there are $\binom{n}{a+b}$ places to position $A \cup B$. Then there are $a! b!$ that place A as the first a of these $a + b$ places. Finally, there are $(n - a - b)!$ ways of ordering the remaining elements not in $A \cup B$.

Thus

$$\begin{aligned} \Pr(\mathcal{E}(A_i, B_i)) &= \frac{n!}{(|A_i| + |B_i|)! (n - |A_i| - |B_i|)!} |A_i|! |B_i|! (n - |A_i| - |B_i|)! \frac{1}{n!} \\ &= \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}. \end{aligned}$$

But then the disjointness of the collection of events $\mathcal{E}(A_i, B_i)$ implies that

$$\sum_{i=1}^m \Pr(\mathcal{E}(A_i, B_i)) \leq 1.$$

2. Let x_1, x_2, \dots, x_n be real numbers such that $x_i \geq 1$ for $i = 1, 2, \dots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^n \varepsilon_i x_i$, ($\varepsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J .
(Hint: use Sperner's lemma.)

Solution: For $A \subseteq [n]$ let $x_A = \sum_{i \in A} x_i - \sum_{i \notin A} x_i$. Let $\mathcal{A} = \{A : x_A \in J\}$. It is enough to show that \mathcal{A} is a Sperner family. Indeed, if $A, B \in \mathcal{A}$ and $A \subset B$ then $x_B - x_A = 2 \sum_{i \in B \setminus A} x_i \geq 2$. Thus we cannot have both $x_A, x_B \in J$.

3. Suppose that we two-color the edges of K_6 Red and Blue. Show that there are at least two monochromatic triangles.

Solution: Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining x to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.