21-301 Combinatorics
Homework 4
Due: Wednesday, October 6

1. Let $s_1, s_2, \ldots, s_m$ be binary strings such that no string is a prefix of another string. ($a = a_1a_2\cdots a_p$ is a prefix of $b = b_1b_2\cdots b_q$ if $p \leq q$ and $a_i = b_i$ for $1 \leq i \leq p$).

   Show that
$$\sum_{i=1}^{m} 2^{-|s_i|} \leq 1$$
   where $|s|$ is the length of string s.

   (Hint: Let $n = \max\{|s_i| : 1 \leq i \leq n\}$. Let x be a random binary string of length n. Consider the events $\mathcal{E}_i = \{s_i$ is a prefix of x.)

   Solution The events $\mathcal{E}_i$ of the hint are *disjoint*. This follows from the assumption that no string is a prefix of another. Thus
$$1 \geq \sum_{i=1}^{m} \Pr(\mathcal{E}_i) = \sum_{i=1}^{m} 2^{-|s_i|}.$$

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every v and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Use the local lemma to prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)

   Solution: Assume that each list $S(v)$ is of size exactly $10d$. Randomly color each vertex v with a color c_v from its list $S(v)$. For each edge $e = \{v, w\}$ and color $c \in S(v) \cap S(w)$ we let $\mathcal{E}_{e,c}$ be the event that $c_v = c_w = c$. Thus $P(\mathcal{E}_{e.c}) = 1/(10d)^2$.

   Note that $\mathcal{E}_{\{v,w\},c}$ depends only on the colors assigned to v and w, and is thus independent of $\mathcal{E}_{\{v',w'\},c'}$ if $\{v', w'\} \cap \{v, w\} = \emptyset$. Hence $\mathcal{E}_{\{v,w\},c}$ only depends on other edges involving v or w. Now there are at most $10d \times d$ events $\mathcal{E}_{\{v,w'\},c'}$ where $c' \in S(v) \cap S(w')$. So the maximum degree in the dependency graph is at most $20d^2$. The result follows from $4 \times 20d^2 \times 1/(10d)^2 < 1$.

3. Show that if $4nk2^{1-k} < 1$ then one can 2-color the integers $1, 2, \ldots, n$ such that there is no mono-colored arithmetic progression of length k. (An arithmetic progression of length k is a set $\{a, a + d, \ldots, a + (k - 1)d\}$.)

   Solution: Color the integers randomly. For an arithmetic progression $S = \{a, a + d, \ldots, a + (k - 1)d\}$ of length k, let $\mathcal{E}_S$ denote the event that S is mono-coloreed. Then $\Pr(\mathcal{E}_S) = 2^{-(k-1)}$.

   Now consider the dependency graph of these events. $\mathcal{E}_S, \mathcal{E}_T$ are independent if S, T are disjoint. A fixed progression S intersects at most kn others: choose $x \in S$ in k ways and then choose d in at most n ways. Now apply the Local Lemma.