21-301 Combinatorics
Homework 4
Due: Wednesday, October 6

1. Let $s_1, s_2, \ldots, s_m$ be binary strings such that no string is a prefix of another string. ($a = a_1a_2 \cdots a_p$ is a prefix of $b = b_1b_2 \cdots b_q$ if $p \leq q$ and $a_i = b_i$ for $1 \leq i \leq p$).

   Show that
   $$\sum_{i=1}^{m} 2^{-|s_i|} \leq 1$$
   where $|s|$ is the length of string s.

   (Hint: Let $n = \max\{|s_i| : 1 \leq i \leq n\}$. Let x be a random binary string of length n. Consider the events $\mathcal{E}_i = \{s_i$ is a prefix of x.)

2. Let $G = (V, E)$ be a graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose that for every v and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Use the local lemma to prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$. (By proper we mean that adjacent vertices get distinct colors.)

3. Show that if $4nk2^{1-k} < 1$ then one can 2-color the integers $1, 2, \ldots, n$ such that there is no mono-colored arithmetic progression of length k.
   (An arithmetic progression of length k is a set $\{a, a + d, \ldots, a + (k - 1)d\}$.)