21-301 Combinatorics
Homework 2
Due: Wednesday, September 15

1. (a) How many strings of length n consisting of 0's and 1's have no two consecutive 1's?
   Solution:
   i. Let a_n be the number of strings made of zeros and ones with no two consecutive ones. If a_n ends in a 0, we have a_{n-1} possible strings. If a_n ends in a 1, it must end in a 01, so we have a_{n-2} possible strings. So,
   $$a_n = a_{n-1} + a_{n-2}.$$
   There is one empty valid sequence, two valid sequences of length 1 and three of length 2.
   Therefore $a_n = F_{n+1}$, where F_n is the n'th Fibonacci number.
   ii. How many strings of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?
   [Hint: reduce the question to (a).]

   (b) How many strings of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?
   Solution
   Let $\{a_n\}$ be a string of length n that satisfies the condition in the problem. Define $\{b_{n-1}\}$ as follows: $b_i = 1$ iff $a_i = a_{i+1}$ and 0 otherwise. The string $\{b_{n-1}\}$ has no two consecutive ones. From (a) above, there are F_n strings of the defined type. For each string b_{n-1} there are 2 strings a_n. So, the answer is $2F_n$.

2. Find a_n if
$$a_n = 6a_{n-1} + 7a_{n-2}, a_0 = 2, a_1 = 10.$$
Let $F(x) = \sum a_n x^n$. Then
$$F(x) - 2 - 10x = 6x(F(x) - 2) + 7x^2F(x),$$
$$F(x) = \frac{2 - 2x}{1 - 6x - 7x^2},$$
$$F(x) = \frac{3/2}{1 - 7x} + \frac{1/2}{1 + x}.$$
So
$$a_n = \frac{3}{2}7^n + \frac{1}{2}(-1)^n.$$

3. Let $S_0 = 1$ and S_n denote the number of ways that $2n$ people sitting in a cycle can shake hands without crossing arms?
   (a) Prove that $S_n = \sum_{i=1}^{n} S_{i-1}S_{n-i}$.
   (b) Deduce that $S_n = \frac{1}{n+1}\binom{2n}{n}$.

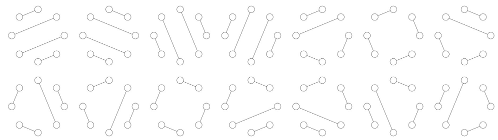

Figure 1: All handshake configurations for 8 people

Solution:

(a) Assume we have a configuration of non-crossing handshakes. Suppose person 1 shakes hands with person k. Then, persons $2, \dots k-1$ must form a configuration of non-crossing handshakes. So. k must be even. Let $k = 2i$. Also, persons $k+1, k+2, \dots 2n$ must form a configuration of non-crossing handshakes. So, the number of configurations of non-crossing handshakes in which person 1 and person $2i$ shake hands is $S_{i-1}S_{n-i}$. Person 1 must shake hands with somebody, so $S_n = \sum_{i=1}^{n} S_{i-1}S_{n-i}$.

(b) Now compare this to the number a_n of ways of triangulating P_{n+1}. Here we have $a_0 = 0, a_1 = a_2 = 1$ and $a_n = \sum_{i=0}^{n} a_i a_{n-i} = \sum_{i=1}^{n-1} a_i a_{n-i}$ for $n \geq 2$ and $a_n = \frac{1}{n}\binom{2n-2}{n-1}$. The claimed value for S_n follows from $S_n = a_{n+1}$. We check this by induction. We let $S_0 = 1 = a_1$ and $S_1 = 1 = a_2$ and then inductively

$$S_n = \sum_{i=1}^{n} S_{i-1}S_{n-i} = \sum_{i=1}^{n} a_i a_{n-i+1} = a_{n+1}.$$