21-301 Combinatorics
Homework 1
Due: Wednesday, September 8

1. How many sequences $(a_1, a_2, \ldots, a_m) \in [n]^m$ satisfy $a_1 < a_2 < \cdots < a_m$? How many satisfy $a_1 \leq a_2 \leq \cdots \leq a_m$?

   Solution: The sequence $a_1 < a_2 < \cdots < a_m$ defines a subset $\{a_1, a_2, \ldots, a_m\}$ of $[n]$ and conversely, every subset of size m defines a sequence after ordering the elements. Thus the answer to the first part is $\binom{n}{m}$.

   For the second part, let $a_0 = 0$ and $a_{m+1} = n$ and let $b_i = a_i - a_{i-1}$ for $i = 1, 2, \ldots, m+1$. There is a 1-1 correspondence between the sequences $a_0 = 0, a_1, a_2, \ldots, a_m, a_{m+1} = n$ and the sequences $b_1, b_2, \ldots, b_{m+1}$ since we can recover $a_i = b_1 + b_2 + \cdots + b_i$ for $i = 1, 2, \ldots, m$. Now we have $b_1 \geq 1$ and $b_i \geq 0, i = 2, 3, \ldots, m+1$ and $b_1 + b_2 + \cdots + b_{m+1} = n$ and any sequence $b_1, b_2, \ldots, b_{m+1}$ with these properties gives rise to a sequence $a_1 \leq a_2 \leq \cdots \leq a_m$. Thus there are $\binom{(n-1)+(m+1)-1}{m} = \binom{n+m-1}{m}$ such sequences.

2. Suppose that a round table has $3n$ seats. Suppose that n families arrive consisting of man/woman/child. They are to be seated round the table in triples: adult,child,adult. How many ways of seating the guests are there so that no family sits together as a complete triple of adult,child,adult.

   Solution: This is a variation on the "Probléme des Ménages". Let A_i denote the set of seatings in which family i sit together. Then for $|S| = k$ we have

   $$|A_S| = 3k!2^k(2(n-k))!(n-k)!\binom{n}{k} = 3n!(2(n-k))!2^k$$

   Explanation: 3 choices for where to place the "first" child. Given this, we choose k seats for children in $\binom{n}{k}$ ways. Then we order the children of the k families in $k!$ ways. Then we place their parents next to them in 2^k ways. The we place the remaining children in $(n-k)!$ ways and adults in $(2(n-k))!$ ways.

   Putting it all together we get that the number of seating arrangements is

   $$3n!\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2(n-k))!2^k.$$

3. Suppose that we have $2n$ distinguishable balls. Suppose that there are n colors and that we have 2 balls of each color. How many ways are there of placing the balls into n boxes, two balls per box, so that there are exactly k boxes containing balls of the same color?

   Solution: Let A_i denote the set of allocations in which box i gets two balls of the same color. Arguing as in the notes on Scrambled Allocations, we see that if $|S| = \ell$ then

   $$|A_S| = \frac{(2(n-\ell))!}{2^{n-\ell}} \times \binom{n}{\ell} \times \ell!.$$

   The extra factor $\binom{n}{\ell} \times \ell!$ comes from allocating the colors to the boxes.

Applying the inclusion-exclusion formula for elements in exactly k of the A_i, we get the expression

$$\sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} \frac{(2(n-\ell))!}{2^{n-\ell}} \binom{n}{\ell} \ell!.$$