Covered so far

8/30/2021

Let $\phi(m,n)$ be the number of mappings from $[n]$ to $[m]$.

Theorem

$$\phi(m,n) = m^n$$

Proof By induction on n.

$$\phi(m,0) = 1 = m^0.$$

$$\begin{aligned}\phi(m,n+1) &= m\phi(m,n)\\ &= m \times m^n\\ &= m^{n+1}.\end{aligned}$$

□

$\phi(m,n)$ is also the number of sequences $x_1 x_2 \cdots x_n$ where

Let $\psi(n)$ be the number of subsets of $[n]$.

Theorem

$$\psi(n) = 2^n.$$

Proof (1) By induction on n.

$\psi(0) = 1 = 2^0$.

$$\begin{aligned} &\psi(n+1) \\ &= \#\{\text{sets containing } n+1\} + \#\{\text{sets not containing } n+1\} \\ &= \psi(n) + \psi(n) \\ &= 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

There is a general principle that if there is a 1-1 correspondence between two finite sets A, B then $|A| = |B|$. Here is a use of this principle.

Proof (2).
For $A \subseteq [n]$ define the map $f_A : [n] \to \{0, 1\}$ by

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

f_A is the characteristic function of A.

Distinct A's give rise to distinct f_A's and vice-versa.

Thus $\psi(n)$ is the number of choices for f_A, which is 2^n by Theorem 51. □

Let $\psi_{odd}(n)$ be the number of odd subsets of $[n]$ and let $\psi_{even}(n)$ be the number of even subsets.

Theorem

$$\psi_{odd}(n) = \psi_{even}(n) = 2^{n-1}.$$

Proof For $A \subseteq [n-1]$ define

$$A' = \begin{cases} A & |A| \text{ is odd} \\ A \cup \{n\} & |A| \text{ is even} \end{cases}$$

The map $A \to A'$ defines a bijection between $[n-1]$ and the odd subsets of $[n]$. So $2^{n-1} = \psi(n-1) = \psi_{odd}(n)$. Futhermore,

$$\psi_{even}(n) = \psi(n) - \psi_{odd}(n) = 2^n - 2^{n-1} = 2^{n-1}.$$

Let $\phi_{1-1}(m,n)$ be the number of 1-1 mappings from $[n]$ to $[m]$.

Theorem

$$\phi_{1-1}(m,n) = \prod_{i=0}^{n-1}(m-i). \tag{1}$$

Proof Denote the RHS of (1) by $\pi(m,n)$. If $m < n$ then $\phi_{1-1}(m,n) = \pi(m,n) = 0$. So assume that $m \geq n$. Now we use induction on n.

If $n = 0$ then we have $\phi_{1-1}(m,0) = \pi(m,0) = 1$.

In general, if $n < m$ then

$$\begin{aligned} \phi_{1-1}(m,n+1) &= (m-n)\phi_{1-1}(m,n) \\ &= (m-n)\pi(m,n) \\ &= \pi(m,n+1). \end{aligned}$$

□

$\phi_{1-1}(m, n)$ also counts the number of length n ordered sequences distinct elements taken from a set of size m.

$$\phi_{1-1}(n, n) = n(n-1)\cdots 1 = n!$$

is the number of ordered sequences of $[n]$ i.e. the number of permutations of $[n]$.

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$$

Let X be a finite set and let

$\binom{X}{k}$ denote the collection of k-subsets of X.

Theorem

$$\left|\binom{X}{k}\right| = \binom{|X|}{k}.$$

Proof Let $n = |X|$,

$$k!\left|\binom{X}{k}\right| = \phi_{1-1}(n,k) = n(n-1)\cdots(n-k+1).$$

Let m, n be non-negative integers. Let Z_+ denote the non-negative integers. Let

$$S(m,n) = \{(i_1, i_2, \ldots, i_n) \in Z_+^n : \; i_1 + i_2 + \cdots + i_n = m\}.$$

Theorem

$$|S(m,n)| = \binom{m+n-1}{n-1}.$$

Proof imagine $m+n-1$ points in a line. Choose positions $p_1 < p_2 < \cdots < p_{n-1}$ and color these points red. Let $p_0 = 0,\ p_n = m+1$. The gap sizes between the red points

$$i_t = p_t - p_{t-1} - 1,\ t = 1, 2, \ldots, n$$

form a sequence in $S(m,n)$ and vice-versa. □

$|S(m,n)|$ is also the number of ways of coloring m *indistinguishable* balls using n colors.

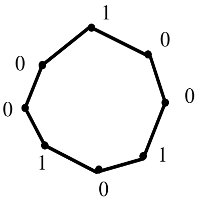
Suppose that we want to count the number of ways of coloring these balls so that each color appears at least once i.e. to compute $|S(m,n)^*|$ where, if $N = \{1, 2, \ldots, \}$

$$S(m,n)^* =$$
$$\{(i_1, i_2, \ldots, i_n) \in N^n : \ i_1 + i_2 + \cdots + i_n = m\}$$
$$= \{(i_1 - 1, i_2 - 1, \ldots, i_n - 1) \in Z_+^n :$$
$$(i_1 - 1) + (i_2 - 1) + \cdots + (i_n - 1) = m - n\}$$

Thus,

$$|S(m,n)^*| = \binom{m-n+n-1}{n-1} = \binom{m-1}{n-1}.$$

Seperated 1's on a cycle



How many ways (patterns) are there of placing k 1's and $n-k$ 0's at the vertices of a polygon with n vertices so that no two 1's are adjacent?
Choose a vertex v of the polygon in n ways and then place a 1 there. For the remainder we must choose $a_1, \ldots, a_k \geq 1$ such that $a_1 + \cdots + a_k = n-k$ and then go round the cycle (clockwise) putting a_1 0's followed by a 1 and then a_2 0's followed by a 1 etc..

Each pattern π arises k times in this way. There are k choices of v that correspond to a 1 of the pattern. Having chosen v there is a unique choice of $a_1, a_2, \ldots, a_k$ that will now give π.

There are $\binom{n-k-1}{k-1}$ ways of choosing the a_i and so the answer to our question is

$$\frac{n}{k}\binom{n-k-1}{k-1.}$$

Theorem

Symmetry

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof Choosing r elements to include is equivalent to choosing $n-r$ elements to exclude. □

Theorem

Pascal's Triangle

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof A $k+1$-subset of $[n+1]$ either
(i) includes $n+1$ —— $\binom{n}{k}$ choices or
(ii) does not include $n+1$ —— $\binom{n}{k+1}$ choices.

Pascal's Triangle

The following array of binomial coefficents, constitutes the famous triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
. . .

Theorem

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \qquad (2)$$

Proof 1: Induction on n for arbitrary k.
Base case: $n = k$; $\binom{k}{k} = \binom{k+1}{k+1}$
Inductive Step: assume true for $n \geq k$.

$$\begin{aligned} \sum_{m=k}^{n+1} \binom{m}{k} &= \sum_{m=k}^{n} \binom{m}{k} + \binom{n+1}{k} \\ &= \binom{n+1}{k+1} + \binom{n+1}{k} && \text{Induction} \\ &= \binom{n+2}{k+1}. && \text{Pascal's triangle} \end{aligned}$$

Proof 2: Combinatorial argument.
If S denotes the set of $k+1$-subsets of $[n+1]$ and S_m is the set of $k+1$-subsets of $[n+1]$ which have largest element $m+1$ then

- $S_k, S_{k+1}, \ldots, S_n$ is a partition of S.
- $|S_k| + |S_{k+1}| + \cdots + |S_n| = |S|$.
- $|S_m| = \binom{m}{k}$.

□

Theorem

Vandermonde's Identity

$$\sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r}=\binom{m+n}{k}.$$

Proof Split $[m+n]$ into $A=[m]$ and $B=[m+n]\setminus[m]$. Let S denote the set of k-subsets of $[m+n]$ and let $S_r=\{X\in S:\ |X\cap A|=r\}$. Then

- $S_0, S_1, \ldots, S_k$ is a partition of S.
- $|S_0|+|S_1|+\cdots+|S_k|=|S|$.
- $|S_r|=\binom{m}{r}\binom{n}{k-r}$.
- $|S|=\binom{m+n}{k}$.

□

Theorem

Binomial Theorem

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Proof Coefficient x^r in $(1+x)(1+x)\cdots(1+x)$: choose x from r brackets and 1 from the rest. □

Applications of Binomial Theorem

- $x = 1$:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

LHS counts the number of subsets of all sizes in $[n]$.

- $x = -1$:

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = (1-1)^n = 0,$$

i.e.

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

and number of subsets of even cardinality = number of subsets of odd cardinality.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Differentiate both sides of the Binomial Theorem w.r.t. x.

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}.$$

Now put $x = 1$.

9/1/2021

Inclusion-Exclusion

2 sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

So if $A_1, A_2 \subseteq A$ and $\overline{A}_i = A \setminus A_i$, $i = 1, 2$ then

$$|\overline{A}_1 \cap \overline{A}_2| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3 sets:

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= |A| - |A_1| - |A_2| - |A_3| \\ &\quad + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

General Case

$A_1, A_2, \ldots, A_N \subseteq A$ and each $x \in A$ has a weight w_x. (In our examples $w_x = 1$ for all x and so $w(X) = |X|$.)

For $S \subseteq [N]$, $A_S = \bigcap_{i \in S} A_i$ and $w(S) = \sum_{x \in S} w_x$.

E.g. $A_{\{4,7,18\}} = A_4 \cap A_7 \cap A_{18}$.

$A_\emptyset = A$.

Inclusion-Exclusion Formula:

$$w\left(\bigcap_{i=1}^{N} \overline{A}_i\right) = \sum_{S \subseteq [N]} (-1)^{|S|} w(A_S).$$

Simple example. How many integers in [1000] are not divisible by 5,6 or 8 i.e. what is the size of $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$ below? Here we take $w_x = 1$ for all x.

$$
\begin{aligned}
A = A_\emptyset &= \{1, 2, 3, \ldots, \} & |A| &= 1000 \\
A_1 &= \{5, 10, 15, \ldots, \} & |A_1| &= 200 \\
A_2 &= \{6, 12, 18, \ldots, \} & |A_2| &= 166 \\
A_3 &= \{8, 16, 24, \ldots, \} & |A_2| &= 125 \\
A_{\{1,2\}} &= \{30, 60, 90, \ldots, \} & |A_{\{1,2\}}| &= 33 \\
A_{\{1,3\}} &= \{40, 80, 120, \ldots, \} & |A_{\{1,3\}}| &= 25 \\
A_{\{2,3\}} &= \{24, 48, 72, \ldots, \} & |A_{\{2,3\}}| &= 41 \\
A_{\{1,2,3\}} &= \{120, 240, 360, \ldots, \} & |A_{\{1,2,3\}}| &= 8
\end{aligned}
$$

$$
\begin{aligned}
|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= 1000 - (200 + 166 + 125) \\
&\qquad + (33 + 25 + 41) - 8 \\
&= 600.
\end{aligned}
$$

Derangements

A derangement of $[n]$ is a permutation π such that

$$\pi(i) \neq i : \ i = 1, 2, \ldots, n.$$

We must express the set of derangements D_n of $[n]$ as the intersection of the complements of sets.
We let $A_i = \{\text{permutations } \pi : \ \pi(i) = i\}$ and then

$$|D_n| = \left| \bigcap_{i=1}^{n} \overline{A}_i \right|.$$

We must now compute $|A_S|$ for $S \subseteq [n]$.

$|A_1| = (n-1)!$: after fixing $\pi(1) = 1$ there are $(n-1)!$ ways of permuting $2, 3, \ldots, n$.

$|A_{\{1,2\}}| = (n-2)!$: after fixing $\pi(1) = 1, \pi(2) = 2$ there are $(n-2)!$ ways of permuting $3, 4, \ldots, n$.

In general

$$|A_S| = (n - |S|)!$$

$$
\begin{aligned}
|D_n| &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\
&= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.
\end{aligned}
$$

When n is large,

$$
\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \approx e^{-1}.
$$

Proof of inclusion-exclusion formula

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) = \begin{cases} 1 & x \in \bigcap_{i=1}^{N} \overline{A}_i \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} w\left(\bigcap_{i=1}^{N} \overline{A}_i\right) &= \sum_{x\in A} w_x (1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N}) \\ &= \sum_{x\in A} w_x \sum_{S\subseteq[N]} (-1)^{|S|} \prod_{i\in S} \theta_{x,i} \\ &= \sum_{S\subseteq[N]} (-1)^{|S|} \sum_{x\in A} w_x \prod_{i\in S} \theta_{x,i} \\ &= \sum_{S\subseteq[N]} (-1)^{|S|} w(A_S). \end{aligned}$$

Euler's Function $\phi(n)$.

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n, other than 1.

$\phi(12) = 4$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of n.

$$A_i = \{x \in [n] : \ p_i \text{ divides } x\}, \qquad 1 \leq i \leq k.$$

$$\phi(n) = \left| \bigcap_{i=1}^{k} \overline{A}_i \right|$$

$$|A_S| = \frac{n}{\prod_{i \in S} p_i} \qquad S \subseteq [k].$$

$$\begin{aligned}
\phi(n) &= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} \\
&= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)
\end{aligned}$$

Surjections

Fix n, m. Let

$$A = \{f : [n] \to [m]\}$$

Thus $|A| = m^n$. Let

$$F(n,m) = \{f \in A : \; f \text{ is onto } [m]\}.$$

How big is $F(n,m)$?
Let

$$A_i = \{f \in F : \; f(x) \neq i, \forall x \in [n]\}.$$

Then

$$F(n,m) = \bigcap_{i=1}^{m} \overline{A}_i.$$

For $S \subseteq [m]$

$$\begin{aligned} A_S &= \{f \in A : \ f(x) \notin S, \forall x \in [n]\}. \\ &= \{f : [n] \to [m] \setminus S\}. \end{aligned}$$

So

$$|A_S| = (m - |S|)^n.$$

Hence

$$\begin{aligned} F(n, m) &= \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n. \end{aligned}$$

Scrambled Allocations

We have n boxes $B_1, B_2, \ldots, B_n$ and $2n$ distinguishable balls $b_1, b_2, \ldots, b_{2n}$.
An allocation of balls to boxes, two balls to a box, is said to be *scrambled* if there does **not** exist i such that box B_i contains balls b_{2i-1}, b_{2i}. Let σ_n be the number of scrambled allocations.

Let A_i be the set of allocations in which box B_i contains b_{2i-1}, b_{2i}. We show that

$$|A_S| = \frac{(2(n-|S|))!}{2^{n-|S|}}.$$

Inclusion-Exclusion then gives

$$\sigma_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2(n-k))!}{2^{n-k}}.$$

First consider $A_\emptyset$:

Each permutation π of $[2n]$ yields an allocation of balls, placing $b_{\pi(2i-1)}, b_{\pi(2i)}$ into box B_i, for $i = 1, 2, \ldots, n$. The order of balls in the boxes is immaterial and so each allocation comes from exactly 2^n distinct permutations, giving

$$|A_\emptyset| = \frac{(2n)!}{2^n}.$$

To get the formula for $|A_S|$ observe that the contents of $2|S|$ boxes are fixed and so we are in essence dealing with $n - |S|$ boxes and $2(n - |S|)$ balls.

9/3/2021

Probléme des Ménages

In how many ways M_n can n male-female couples be seated around a table, alternating male-female, so that no person is seated next to their partner?

Let A_i be the set of seatings in which couple i sit together.

If $|S| = k$ then

$$|A_S| = 2k!(n-k)!^2 \times d_k.$$

d_k is the number of ways of placing k 1's on a cycle of length $2n$ so that no two 1's are adjacent. (We place a person at each 1 and his/her partner on the succeeding 0).

2 choices for which seats are occupied by the men or women. $k!$ ways of assigning the couples to the positions; $(n-k)!^2$ ways of assigning the rest of the people.

$$d_k = \frac{2n}{k}\binom{2n-k-1}{k-1} = \frac{2n}{2n-k}\binom{2n-k}{k}.$$

(See slides 11 and 12).

$$\begin{aligned} M_n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k} \times 2k!(n-k)!^2 \times \frac{2n}{2n-k}\binom{2n-k}{k} \\ &= 2n!\sum_{k=0}^{n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!. \end{aligned}$$

The weight of elements in exactly k sets:

Observe that

$$\prod_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i}) = 1 \text{ iff } x \in A_i, i \in S \text{ and } x \notin A_i, i \notin S.$$

W_k is the total weight of elements in exactly k of the A_i:

$$\begin{aligned}
N_k &= \sum_{x \in A} w_x \sum_{|S|=k} \prod_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i}) \\
&= \sum_{|S|=k} \sum_{x \in A} w_x \prod_{i \in S} \theta_{x,i} \prod_{i \notin S} (1 - \theta_{x,i}) \\
&= \sum_{|S|=k} \sum_{T \supseteq S} \sum_{x \in A} w_x (-1)^{|T \setminus S|} \prod_{i \in T} \theta_{x,i} \\
&= \sum_{|S|=k} \sum_{T \supseteq S} (-1)^{|T \setminus S|} w(A_T) \\
&= \sum_{\ell=k}^{N} \sum_{|T|=\ell} (-1)^{\ell-k} \binom{\ell}{k} w(A_T).
\end{aligned}$$

As an example. Let $D_{n,k}$ denote the number of permutations π of $[n]$ for which there are exactly k indices i for which $\pi(i) = i$. Then

$$
\begin{aligned}
D_{n,k} &= \sum_{\ell=k}^{n} \binom{n}{\ell} (-1)^{\ell-k} \binom{\ell}{k} (n-\ell)! \\
&= \sum_{\ell=k}^{n} \frac{n!}{\ell!(n-\ell)!} (-1)^{\ell-k} \frac{\ell!}{k!(\ell-k)!} (n-\ell)! \\
&= \frac{n!}{k!} \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{(\ell-k)!} \\
&= \frac{n!}{k!} \sum_{r=0}^{n-k} \frac{(-1)^{r}}{r!} \\
&\approx \frac{n!}{ek!}
\end{aligned}
$$

when n is large and k is constant.

9/8/2021

Recurrence Relations

Suppose $a_0, a_1, a_2, \ldots, a_n, \ldots,$is an infinite sequence.
A recurrence recurrence relation is a set of equations

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_{n-k}). \tag{3}$$

The whole sequence is determined by (18) and the values of $a_0, a_1, \ldots, a_{k-1}$.

Linear Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \qquad n \geq 2.$$

$a_0 = a_1 = 1$.

$b_n = |B_n| = |\{x \in \{a, b, c\}^n : aa \text{ does not occur in } x\}|.$

$b_1 = 3 :$ $a\ b\ c$

$b_2 = 8 :$ $ab\ ac\ ba\ bb\ bc\ ca\ cb\ cc$

$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2.$$

Let

$$B_n = B_n^{(b)} \cup B_n^{(c)} \cup B_n^{(a)}$$

where $B_n^{(\alpha)} = \{x \in B_n : \ x_1 = \alpha\}$ for $\alpha = a, b, c$.

Now $|B_n^{(b)}| = |B_n^{(c)}| = |B_{n-1}|$. The map $f : B_n^{(b)} \to B_{n-1}$,

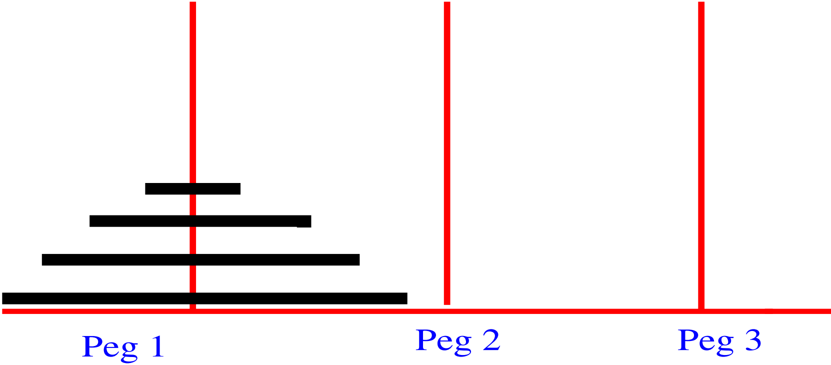
$$f(bx_2x_3 \dots x_n) = x_2x_3 \dots x_n \text{ is a bijection.}$$

$B_n^{(a)} = \{x \in B_n : \ x_1 = a$ and $x_2 = b$ or $c\}$. The map $g : B_n^{(a)} \to B_{n-1}^{(b)} \cup B_{n-1}^{(c)}$,

$$g(ax_2x_3 \dots x_n) = x_2x_3 \dots x_n \text{ is a bijection.}$$

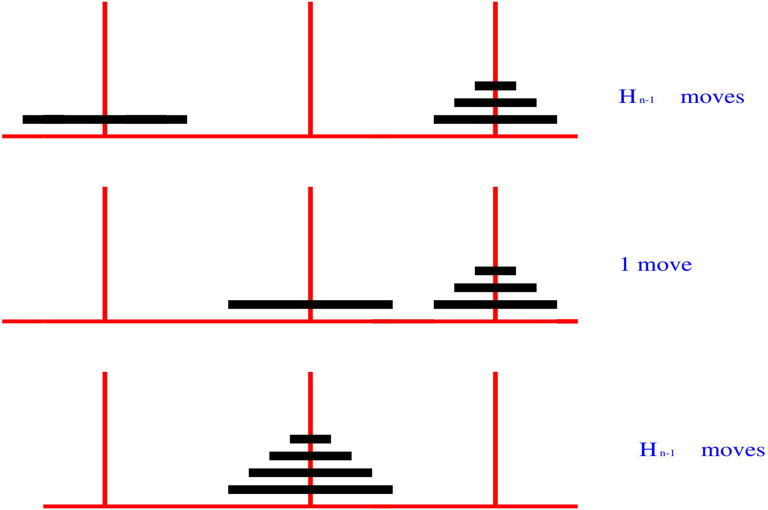
Hence, $|B_n^{(a)}| = 2|B_{n-2}|$.

Towers of Hanoi



H_n is the minimum number of moves needed to shift n rings from Peg 1 to Peg 2. One is not allowed to place a larger ring on top of a smaller ring.

xxx



We see that $H_1 = 1$ and $H_n = 2H_{n-1} + 1$ for $n \geq 2$.

So,

$$\frac{H_n}{2^n} - \frac{H_{n-1}}{2^{n-1}} = \frac{1}{2^n}.$$

Summing these equations give

$$\frac{H_n}{2^n} - \frac{H_1}{2} = \frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{4} = \frac{1}{2} - \frac{1}{2^n}.$$

So

$$H_n = 2^n - 1.$$

A has n dollars. Everyday A buys one of a Bun (1 dollar), an Ice-Cream (2 dollars) or a Pastry (2 dollars). How many ways are there (sequences) for A to spend his money?
Ex. BBPIIPBI represents "Day 1, buy Bun. Day 2, buy Bun etc.".

$$\begin{aligned} u_n &= \text{number of ways} \\ &= u_{n,B} + u_{n,I} + u_{n,P} \end{aligned}$$

where $u_{n,B}$ is the number of ways where A buys a Bun on day 1 etc.
$u_{n,B} = u_{n-1}, \; u_{n,I} = u_{n,P} = u_{n-2}$.
So

$$u_n = u_{n-1} + 2u_{n-2},$$

and

$$u_0 = u_1 = 1.$$

If $a_0, a_1, \ldots, a_n$ is a sequence of real numbers then its **(ordinary) generating function** $a(x)$ is given by

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$$

and we write

$$a_n = [x^n]a(x).$$

For more on this subject see Generatingfunctionology by the late Herbert S. Wilf. The book is available from https://www.math.upenn.edu// wilf/DownldGF.html

$a_n = 1$

$$a(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$a_n = n + 1$.

$$a(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

$a_n = n$.

$$a(x) = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$$

Generalised binomial theorem:

$a_n = \binom{\alpha}{n}$

$$a(x) = (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

$a_n = \binom{m+n-1}{n}$

$$a(x) = \frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n.$$

General view.

Given a recurrence relation for the sequence (a_n), we

(a) Deduce from it, an equation satisfied by the generating function $a(x) = \sum_n a_n x^n$.

(b) Solve this equation to get an explicit expression for the generating function.

(c) Extract the coefficient a_n of x^n from $a(x)$, by expanding $a(x)$ as a power series.

Solution of linear recurrences

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \qquad n \geq 2.$$

$a_0 = 1, a_1 = 9$.

$$\sum_{n=2}^{\infty}(a_n - 6a_{n-1} + 9a_{n-2})x^n = 0. \tag{4}$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= a(x) - a_0 - a_1 x \\
&= a(x) - 1 - 9x. \\
\sum_{n=2}^{\infty} 6a_{n-1} x^n &= 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} \\
&= 6x(a(x) - a_0) \\
&= 6x(a(x) - 1). \\
\sum_{n=2}^{\infty} 9a_{n-2} x^n &= 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 9x^2 a(x).
\end{aligned}
$$

$$a(x) - 1 - 9x - 6x(a(x) - 1) + 9x^2 a(x) = 0$$

or

$$a(x)(1 - 6x + 9x^2) - (1 + 3x) = 0.$$

$$\begin{aligned} a(x) &= \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(1-3x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + 3x\sum_{n=0}^{\infty}(n+1)3^n x^n \\ &= \sum_{n=0}^{\infty}(n+1)3^n x^n + \sum_{n=0}^{\infty} n3^n x^n \\ &= \sum_{n=0}^{\infty}(2n+1)3^n x^n. \end{aligned}$$

$$a_n = (2n+1)3^n.$$

Fibonacci sequence:

$$\sum_{n=2}^{\infty}(a_n - a_{n-1} - a_{n-2})x^n = 0.$$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(a(x) - a_0 - a_1 x) - (x(a(x) - a_0)) - x^2 a(x) = 0.$$

$$a(x) = \frac{1}{1 - x - x^2}.$$

$$
\begin{aligned}
a(x) &= -\frac{1}{(\xi_1 - x)(\xi_2 - x)} \\
&= \frac{1}{\xi_1 - \xi_2}\left(\frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x}\right) \\
&= \frac{1}{\xi_1 - \xi_2}\left(\frac{\xi_1^{-1}}{1 - x/\xi_1} - \frac{\xi_2^{-1}}{1 - x/\xi_2}\right)
\end{aligned}
$$

where

$$\xi_1 = -\frac{\sqrt{5}+1}{2} \text{ and } \xi_2 = \frac{\sqrt{5}-1}{2}$$

are the 2 roots of

$$x^2 + x - 1 = 0.$$

Therefore,

$$\begin{aligned} a(x) &= \frac{\xi_1^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_1^{-n} x^n - \frac{\xi_2^{-1}}{\xi_1 - \xi_2} \sum_{n=0}^{\infty} \xi_2^{-n} x^n \\ &= \sum_{n=0}^{\infty} \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} x^n \end{aligned}$$

and so

$$\begin{aligned} a_n &= \frac{\xi_1^{-n-1} - \xi_2^{-n-1}}{\xi_1 - \xi_2} \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{\sqrt{5}+1}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right). \end{aligned}$$

9/10/2021

Inhomogeneous problem

$$a_n - 3a_{n-1} = n^2 \qquad n \geq 1.$$

$a_0 = 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= \sum_{n=1}^{\infty} n^2 x^n \\
\sum_{n=1}^{\infty} n^2 x^n &= \sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^n \\
&= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \\
&= \frac{x+x^2}{(1-x)^3} \\
\sum_{n=1}^{\infty}(a_n - 3a_{n-1})x^n &= a(x) - 1 - 3xa(x) \\
&= a(x)(1-3x) - 1.
\end{aligned}$$

$$\begin{aligned} a(x) &= \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x} \\ &= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D+1}{1-3x} \end{aligned}$$

where

$$\begin{aligned} x+x^2 &\cong A(1-x)^2(1-3x) + B(1-x)(1-3x) \\ &\qquad + C(1-3x) + D(1-x)^3. \end{aligned}$$

Then

$$A = -1/2,\ B = 0,\ C = -1,\ D = 3/2.$$

So

$$
\begin{aligned}
a(x) &= \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x} \\
&= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n
\end{aligned}
$$

So

$$
\begin{aligned}
a_n &= -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2}3^n \\
&= -\frac{3}{2} - \frac{3n}{2} - \frac{n^2}{2} + \frac{5}{2}3^n.
\end{aligned}
$$

Products of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n, \; b(x)) = \sum_{n=0}^{\infty} b_n x^n.$$

$$\begin{aligned} a(x)b(x) &= (a_0 + a_1 x + a_2 x^2 + \cdots) \times \\ &\qquad\qquad (b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \\ &\qquad\qquad (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Derangements

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}.$$

Explanation: $\binom{n}{k} d_{n-k}$ is the number of permutations with exactly k cycles of length 1. Choose k elements ($\binom{n}{k}$ ways) for which $\pi(i) = i$ and then choose a derangement of the remaining $n-k$ elements.

So

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \\ \sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{d_{n-k}}{(n-k)!} \right) x^n. \end{aligned} \tag{5}$$

Let

$$d(x) = \sum_{m=0}^{\infty} \frac{d_m}{m!} x^m.$$

From (5) we have

$$\begin{aligned} \frac{1}{1-x} &= e^x d(x) \\ d(x) &= \frac{e^{-x}}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \right) x^n. \end{aligned}$$

So

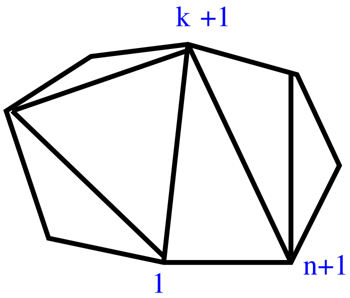
$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Triangulation of n-gon

Let

$$\begin{aligned} a_n &= \text{number of triangulations of } P_{n+1} \\ &= \sum_{k=0}^{n} a_k a_{n-k} \qquad n \geq 2 \end{aligned} \tag{6}$$

$a_0 = 0$, $a_1 = a_2 = 1$.



Explanation of (6):
$a_k a_{n-k}$ counts the number of triangulations in which edge $1, n+1$ is contained in triangle $1, k+1, n+1$.
There are a_k ways of triangulating $1, 2, \ldots, k+1, 1$ and for each such there are a_{n-k} ways of triangulating $k+1, k+2, \ldots, n+1, k+1$.

$$x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n.$$

But,

$$x + \sum_{n=2}^{\infty} a_n x^n = a(x)$$

since $a_0 = 0, a_1 = 1$.

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n \\ &= a(x)^2. \end{aligned}$$

So

$$a(x) = x + a(x)^2$$

and hence

$$a(x) = \frac{1+\sqrt{1-4x}}{2} \text{ or } \frac{1-\sqrt{1-4x}}{2}.$$

But $a(0) = 0$ and so

$$\begin{aligned} a(x) &= \frac{1-\sqrt{1-4x}}{2} \\ &= \frac{1}{2} - \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^{2n-1}} \binom{2n-2}{n-1} (-4x)^n\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n. \end{aligned}$$

So

$$a_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

9/13/2021

Colouring Problem

Theorem

Let $A_1, A_2, \ldots, A_n$ be subsets of A and $|A_i| = k$ for $1 \leq i \leq n$. If $n < 2^{k-1}$ then there exists a partition $A = R \cup B$ such that

$$A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \qquad 1 \leq i \leq n.$$

[R = Red elements and B= Blue elements.]

Proof Randomly colour A.
Ω = $\{R, B\}^A = \{f : A \to \{R, B\}\}$, uniform distribution.

$$BAD = \{\exists i : \ A_i \subseteq R \text{ or } A_i \subseteq B\}.$$

Claim: $\mathbf{Pr}(BAD) < 1$.
Thus $\Omega \setminus BAD \neq \emptyset$ and this proves the theorem.

$$BAD(i) = \{A_i \subseteq R \text{ or } A_i \subseteq B\} \text{ and } BAD = \bigcup_{i=1}^{n} BAD(i).$$

Boole's Inequality: if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$ are a collection of events, then

$$\mathbf{Pr}\left(\bigcup_{i=1}^{N} \mathcal{A}_i\right) \leq \sum_{i=1}^{N} \mathbf{Pr}(\mathcal{A}_i).$$

This easily proved by induction on N. When $N = 2$ we use

$$\mathbf{Pr}(\mathcal{A}_1 \cup \mathcal{A}_2) = \mathbf{Pr}(\mathcal{A}_1) + \mathbf{Pr}(\mathcal{A}_2) - \mathbf{Pr}(\mathcal{A}_1 \cap \mathcal{A}_2) \leq \mathbf{Pr}(\mathcal{A}_1 \cup \mathcal{A}_2).$$

In general,

$$\mathbf{Pr}\left(\bigcup_{i=1}^{N} \mathcal{A}_i\right) \leq \mathbf{Pr}\left(\bigcup_{i=1}^{N-1} \mathcal{A}_i\right) + \mathbf{Pr}(\mathcal{A}_N) \leq \sum_{i=1}^{N-1} \mathbf{Pr}(\mathcal{A}_i) + \mathbf{Pr}(\mathcal{A}_N).$$

The first inequality is the two event case and the second is by induction on N.

So,

$$\begin{aligned}
\mathbf{Pr}(BAD) &\leq \sum_{i=1}^{n} \mathbf{Pr}(BAD(i)) \\
&= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{k-1} \\
&= n/2^{k-1} \\
&< 1.
\end{aligned}$$

Example of system which is not 2-colorable.

Let $n = \binom{2k-1}{k}$ and $A = [2k-1]$ and

$$\{A_1, A_2, \ldots, A_n\} = \binom{[2k-1]}{k}.$$

Then in any 2-coloring of $A_1, A_2, \ldots, A_n$ there is a set A_i all of whose elements are of one color.

Suppose A is partitioned into 2 sets R, B. At least one of these two sets is of size at least k (since $(k-1)+(k-1) < 2k-1$). Suppose then that $R \geq k$ and let S be any k-subset of R. Then there exists i such that $A_i = S \subseteq R$.

Tournaments

n players in a tournament each play each other i.e. there are $\binom{n}{2}$ games.

Fix some k. Is it possible that for every set S of k players there is a person w_S who beats everyone in S?

Suppose that the results of the tournament are decided by a random coin toss.

Fix S, $|S| = k$ and let $\mathcal{E}_S$ be the event that nobody beats everyone in S.

The event

$$\mathcal{E} = \bigcup_S \mathcal{E}_S$$

is that there is a set S for which w_S does not exist.

We only have to show that $\mathbf{Pr}(\mathcal{E}) < 1$.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}) &\le \sum_{|S|=k} \mathbf{Pr}(\mathcal{E}_S) \\
&= \binom{n}{k}(1-2^{-k})^{n-k} \\
&< n^k e^{-(n-k)2^{-k}} \\
&= \exp\{k \ln n - (n-k)2^{-k}\} \\
&\to 0
\end{aligned}
$$

since we are assuming here that k is fixed independent of n.

Graph Crossing Number

The crossing number of a graph G is the minimum number of edge crossings of a drawing of G in the plane.

Euler's forula implies that a planar graph with n vertices has at most $3n$ edges.

This implies that a graph $G = (V, E)$ requires at least $|E| - 3|V|$ crossings.

Theorem

If $|E| > 4|V|$ then G has crossing number $\Omega(|E|^3/|V|^2)$.

If $|E| \approx |V|^{3/2}$ then this gives $\Omega(|V|^{5/2})$ whereas $|E| - 3|V| = O(|V|^{3/2})$.

Proof

Suppose that G has a drawing with k crossings and let $0 < p < 1$.

Let $G_p = (V_p, E_p)$ denote the subgraph of G obtained by including each vertex in V_p independently with probability p.

E_p is then the set of edges $\{x, y\}$ such that $x, y \in V_p$.

$$\mathbf{E}(|V_p|) = p|V| \text{ and } \mathbf{E}(|E_p| = p^2|E|).$$

Also,

$$\mathbf{E}(\text{number of crossings in the drawing of } G_p) = p^4 k.$$

So,

$$p^4 k \geq \mathbf{E}(|E_p| - 3|V_p|) = p^2|E| - 3p|V|.$$

So

$$k \geq \frac{p^2|E| - 3p|V|}{p^4}.$$

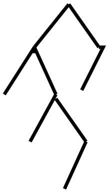
Maximising the RHS over $p \leq 1$ gives $p = 4|V|/|E|$ and

$$k \geq \frac{|E|^3}{64|V|^2}.$$

□

9/15/2021

Random Binary Search Trees



A binary tree consists of a set of *nodes*, one of which is the *root*. Each node is connected to 0,1 or 2 nodes below it and every node other than the root is connected to exactly one node above it. The root is the highest node.

The depth of a node is the number of edges in its path to the root.

The depth of a tree is the maximum over the depths of its nodes.

Starting with a tree T_0 consisting of a single root r, we grow a tree T_n as follows:

The n'th *particle* starts at r and flips a fair coin. It goes left (L) with probability 1/2 and right (R) with probability 1/2.

It tries to move along the tree in the chosen direction. If there is a node below it in this direction then it goes there and continues its random moves. Otherwise it creates a new node where it wanted to move and stops.

Let D_n be the depth of this tree.
Claim: for any $t \geq 0$,

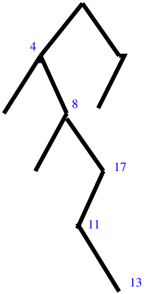
$$\mathbf{Pr}(D_n \geq t) \leq (n2^{-(t-1)/2})^t.$$

Proof The process requires at most n^2 coin flips and so we let $\Omega = \{L, R\}^{n^2}$ – most coin flips will not be needed most of the time.

$$DEEP = \{D_n \geq t\}.$$

For $P \in \{L, R\}^t$ and $S \subseteq [n]$, $|S| = t$ let $DEEP(P, S)$ = {the particles $S = \{s_1, s_2, \ldots, s_t\}$ follow P in the tree i.e. the first i moves of s_i are along P, $1 \leq i \leq t$}.

$$DEEP = \bigcup_P \bigcup_S DEEP(P, S).$$



S={4,8,11,13,17}

t=5 and DEEP(P,S) occurs if
4 goes L...
8 goes LR...
17 goes LRR...
11 goes LRRL...
13 goes LRRLR...

$$
\begin{aligned}
\mathbf{Pr}(DEEP) &\le \sum_P \sum_S \mathbf{Pr}(DEEP(P,S)) \\
&= \sum_P \sum_S 2^{-(1+2+\cdots+t)} \\
&= \sum_P \sum_S 2^{-t(t+1)/2} \\
&= 2^t \binom{n}{t} 2^{-t(t+1)/2} \\
&\le 2^t n^t 2^{-t(t+1)/2} \\
&= (n2^{-(t-1)/2})^t.
\end{aligned}
$$

So if we put $t = A \log_2 n$ then

$$\mathbf{Pr}(D_n \ge A \log_2 n) \le (2n^{1-A/2})^{A \log_2 n}$$

which is very small, for $A > 2$.

A problem with hats

There are n people standing a circle. They are blind-folded and someone places a hat on each person's head. The hat has been randomly colored Red or Blue.

They take off their blind-folds and everyone can see everyone else's hat. Each person then simultaneously declares (i) my hat is red or (ii) my hat is blue or (iii) or I pass.

They win a big prize if the people who opt for (i) or (ii) are all correct. They pay a big penalty if there is a person who incorrectly guesses the color of their hat.

Is there a strategy which means they will win with probability better than 1/2?

Suppose that we partition $Q_n = \{0, 1\}^n$ into 2 sets W, L which have the property that L is a cover i.e. if $x = x_1 x_2 \cdots x_n \in W = Q_n \setminus L$ then there is $y_1 y_2 \cdots y_n \in L$ such that $h(x, y) = 1$ where

$$h(x, y) = |\{j : \ x_j \neq y_j\}|.$$

Hamming distance between x and y.

Assume that $0 \equiv Red$ and $1 \equiv Blue$. Person i knows x_j for $j \neq i$ (color of hat j) and if there is a unique value ξ of x_i which places x in W then person i will declare that their hat has color ξ.

The people assume that $x \in W$ and if indeed $x \in W$ then there is at least one person who will be in this situation and any such person will guess correctly.

Is there a small cover L?

Let $p = \frac{\ln n}{n}$. Choose L_1 randomly by placing $y \in Q_n$ into L_1 with probability p.

Then let L_2 be those $z \in Q_n$ which are not at Hamming distance ≤ 1 from some member of L_1.

Clearly $L = L_1 \cup L_2$ is a cover and
$\mathbf{E}(|L|) = 2^n p + 2^n(1-p)^{n+1} \leq 2^n(p + e^{-np}) \leq 2^n \frac{2 \ln n}{n}$.

So there must exist a cover of size at most $2^n \frac{2 \ln n}{n}$ and the players can win with probability at least $1 - \frac{2 \ln n}{n}$.

9/22/2021

Hoeffding's Inequality – I

Let $X_1, X_2, \ldots, X_n$ be independent random variables taking values such that $\mathbf{Pr}(X_i = 1) = 1/2 = \mathbf{Pr}(X_i = -1)$ for $i = 1, 2, \ldots, n$. Let $X = X_1 + X_2 + \cdots + X_n$. Then for any $t \geq 0$

$$\mathbf{Pr}(|X| \geq t) < 2e^{-t^2/2n}.$$

Proof: For any $\lambda > 0$ we have

$$\begin{aligned} \mathbf{Pr}(X \geq t) &= \mathbf{Pr}(e^{\lambda X} \geq e^{\lambda t}) \\ &\leq e^{-\lambda t}\mathbf{E}(e^{\lambda X}). \end{aligned}$$

Now for $i = 1, 2, \ldots, n$ we have

$$\mathbf{E}(e^{\lambda X_i}) = \frac{e^{-\lambda} + e^{\lambda}}{2} = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots < e^{\lambda^2/2}.$$

So, by independence,

$$\mathbf{E}(e^{\lambda X}) = \mathbf{E}\left(\prod_{i=1}^{n} e^{\lambda X_i}\right) = \prod_{i=1}^{n} \mathbf{E}(e^{\lambda X_i}) \le e^{\lambda^2 n/2}.$$

Hence,

$$\mathbf{Pr}(X \ge t) \le e^{-\lambda t + \lambda^2 n/2}.$$

We choose $\lambda = t/n$ to minimise $-\lambda t + \lambda^2 n/2$. This yields

$$\mathbf{Pr}(X \ge t) \le e^{-t^2/2n}.$$

Similarly,

$$\begin{aligned} \mathbf{Pr}(X \le -t) &= \mathbf{Pr}(e^{-\lambda X} \ge e^{\lambda t}) \\ &\le e^{-\lambda t}\mathbf{E}(e^{-\lambda X}) \\ &\le e^{-\lambda t + \lambda^2 n/2}. \end{aligned}$$

Discrepancy

Suppose that $|X| = n$ and $\mathcal{F} \subseteq \mathcal{P}(X)$. If we color the elements of X with Red and Blue i.e. partition X in $R \cup B$ then the discrepancy $disc(\mathcal{F}, R, B)$ of this coloring is defined

$$disc(\mathcal{F}, R, B) = \max_{F \in \mathcal{F}} disc(F, R, B)$$

where $disc(F, R, B) = ||R \cap F| - |B \cap F||$ i.e. the absolute difference between the number of elements of F that are colored Red and the number that are colored Blue.

Claim:

If $|\mathcal{F}| = m$ then there exists a coloring R, B such that $disc(\mathcal{F}, R, B) \le (2n\log_e(2m))^{1/2}$.

Proof Fix $F \in \mathcal{F}$ and let $s = |F|$. If we color X randomly and let $Z = |R \cap F| - |B \cap F|$ then Z is the sum of s independent ± 1 random variables.

So, by the Hoeffding inequality,

$$\mathbf{Pr}(|Z| \ge (2n\log_e(2m))^{1/2}) < 2e^{-n\log_e(2m)/s} \le \frac{1}{m}.$$

9/24/2021

The Local Lemma

We go back to the coloring problem at the beginning of these slides. We now place a different restriction on the sets involved.

Theorem

Let $A_1, A_2, \ldots, A_n$ be subsets of A where $|A_i| \geq k$ for $1 \leq i \leq n$. If each A_i intersects at most 2^{k-3} other sets then there exists a partition $A = R \cup B$ such that

$$A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \qquad 1 \leq i \leq n.$$

Symmetric Local Lemma: We consider the following situation. $X = \{x_1, x_2, \ldots, x_N\}$ is a collection of independent random variables. Suppose that we have events $\mathcal{E}_i, i = 1, 2, \ldots, m$ where $\mathcal{E}_i$ depends only on the set $X_i \subseteq X$. Thus if $X_i \cap X_j = \emptyset$ then $\mathcal{E}_i$ and $\mathcal{E}_j$ are independent.
The dependency graph Γ has vertex set $[m]$ and an edge (i, j) iff $X_i \cap X_j \neq \emptyset$.

Theorem

Let

$$p = \max_i \mathbf{Pr}(\mathcal{E}_i) \text{ and let } d \text{ be the maximum degree of } \Gamma.$$

$$4dp \leq 1 \text{ implies that } \mathbf{Pr}\left(\bigcap_{i=1}^{m} \bar{\mathcal{E}}_i\right) \geq (1 - 2p)^m > 0.$$

Proof of Theorem 14: We randomly color the elements of A Red and Blue. Let $\mathcal{E}_i$ be the event that A_i is mono-colored. Clearly, $\mathbf{Pr}(\mathcal{E}_i) \leq 2^{-(k-1)}$. Thus,

$$p \leq 2^{-(k-1)}.$$

The degree of vertex i of Γ is the number of j such that $A_i \cap A_j \neq \emptyset$. So, by assumption,

$$d \leq 2^{k-3}.$$

Theorem 15 implies that $\mathbf{Pr}\left(\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i\right) > 0$ and so the required coloring exists.

Theorem

Let $G = (V, E)$ be an r-regular graph. If r is sufficiently large, then E can be partitioned into E_1, E_2 so that if $G_i = (V, E_i), i = 1, 2$ then

$$\frac{r}{2} - (20r \log r)^{1/2} \le \delta(G_i) \le \Delta(G_i) \le \frac{r}{2} + (20r \log r)^{1/2}.$$

Proof: We randomly partition the edges of G by independently placing e into E_1 E_1 with probability 1/2. For $v \in V$, we let $\mathcal{E}_v$ be the event that the degree $d_1(v)$ in G_1 satisfies

$$d_1(v) \notin \left[\frac{r}{2} - (3r \log r)^{1/2}, \frac{r}{2} + (3r \log r)^{1/2}\right].$$

It follows from Hoeffding's Inequality - I with $t = (3r \log r)^{1/2}$ that

$$\mathbf{Pr}(\mathcal{E}_v) \leq 2e^{-t^2/2r} = 2r^{-3/2}. \tag{7}$$

Furthermore, $\mathcal{E}_v$ is independent of the events $\mathcal{E}_w$ for vertices w at distance 2 or more from v in G. Thus,

$$d \leq r.$$

Clearly, $4 \cdot 2r^{-3/2} \cdot r \leq 1$ for r large and the result follows from Theorem 15. I.e. $\mathbf{Pr}\left(\bigcap_{v \in V} \bar{\mathcal{E}}_v\right) > 0$ which imples that there exists a partition where none of the events $\mathcal{E}_v, v \in V$ occur.

9/27/2021

For the next application, let $D = (V, E)$ be a k-regular digraph. By this we mean that each vertex has exactly k in-neighbors and k out-neighbors.

Theorem

Every k-regular digraph has a collection of $\lfloor k/(4 \log k) \rfloor$ vertex disjoint cycles.

Proof: Let $r = \lfloor k/(4 \log k) \rfloor$ and color the vertices of D with colors $[r]$. For $v \in V$, let $\mathcal{E}_v$ be the event that there is a color missing at the out-neighbors of v. We will show that $\mathbf{Pr}\left(\bigcap_{v \in V} \bar{\mathcal{E}}_v\right) > 0$.

Suppose then that none of the events $\mathcal{E}_v, v \in V$ occur. Consider the graph D_j induced by a single color $j \in [r]$. Note that D_j is not the empty graph. Let $P_j = (v_1, v_2, \ldots, v_m)$ be a longest directed path in D_j. Let w be an out-neighbor of v_m of color j. We must have $w \in \{v_1, \ldots, v_m\}$, else P_j is not a longest path in D_j. Thus each $D_j, j \in [r]$ contains a cycle and these cycles are vertex disjoint.

We first estimate

$$\mathbf{Pr}(\mathcal{E}_v) \leq k\left(1-\frac{1}{r}\right)^k \leq ke^{-k/r} \leq ke^{-4\log k} = k^{-3}.$$

On the other hand, if $N^+(v)$ denotes the out-neighbors of v plus v then $\mathcal{E}_v$ is independent of all events $\mathcal{E}_w$ for which $N^+(v) \cap N^+(w) = \emptyset$. It follows that

$$d \leq k^2.$$

To apply Theorem 15 we need to have $4k^{-3}k^2 \leq 1$. This is true for $k \geq 4$. For $k \leq 3$ we have $r = 1$ and the local lemma is not needed.

Let $\mathcal{P}_n = \{A : A \subseteq [n]\}$ denote the *power set* of $[n]$.

$\mathcal{A} \subseteq \mathcal{P}_n$ is a *Sperner* family if $A, B \in \mathcal{A}$ implies that $A \not\subseteq B$ and $B \not\subseteq A$

Theorem

If $\mathcal{A} \subseteq \mathcal{P}_n$ is a Sperner family $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof We will show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1. \tag{8}$$

Now $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for all k and so

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Proof of (8)**:** Let π be a random permutation of $[n]$.

For a set $A \in \mathcal{A}$ let $\mathcal{E}_A$ be the event

$$\{\pi(1), \pi(2), \ldots, \pi(|A|)\} = A.$$

If $A, B \in \mathcal{A}$ then the events $\mathcal{E}_A, \mathcal{E}_B$ are disjoint.

So

$$\sum_{A \in \mathcal{A}} \mathbf{Pr}(\mathcal{E}_A) \leq 1.$$

On the other hand, if $A \in \mathcal{A}$ then

$$\mathbf{Pr}(\mathcal{E}_A) = \frac{|A|!(n - |A|)!}{n!} = \frac{1}{\binom{n}{|A|}}$$

and (18) follows.

The set of all sets of size $\lfloor n/2 \rfloor$ is a Sperner family and so the bound in the above theorem is best possible.

Inequality (8) can be generalised as follows: Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1} \in \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$.

Theorem

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$

Proof Let π be a random permutation of $[n]$.

Let $\mathcal{E}(A)$ be the event $\{\pi(1), \pi(2), \ldots, \pi(|A|) = A\}\}$.

Let

$$Z_i = \begin{cases} 1 & \mathcal{E}(A_i) \text{ occurs.} \\ 0 & \text{otherwise.} \end{cases}$$

and let $Z = \sum_i Z_i$ be the number of events $\mathcal{E}(A_i)$ that occur.

Now our family is such that $Z \leq s$ for all π and so

$$E(Z) = \sum_i E(Z_i) = \sum_i \mathbf{Pr}(\mathcal{E}(A_i)) \leq s.$$

On the other hand, $A \in \mathcal{A}$ implies that $\mathbf{Pr}(\mathcal{E}(A)) = \frac{1}{\binom{n}{|A|}}$ and the required inequality follows. □

9/29/2021

Intersecting Families A family $\mathcal{A} \subseteq \mathcal{P}_n$ is an *intersecting* family if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$.

Theorem

If $\mathcal{A}$ is an intersecting family then $|\mathcal{A}| \leq 2^{n-1}$.

Proof Pair up each $A \in \mathcal{P}_n$ with its complement $A^c = [n] \setminus A$. This gives us 2^{n-1} pairs altogether.
Since $\mathcal{A}$ is intersecting it can contain at most one member of each pair. $\square$

If $\mathcal{A} = \{A \subseteq [n] : 1 \in A\}$ then $\mathcal{A}$ is intersecting and $|\mathcal{A}| = 2^{n-1}$ and so the above theorem is best possible.

Theorem

If $\mathcal{A}$ is an intersecting family and $A \in \mathcal{A}$ implies that $|A| = k \leq \lfloor n/2 \rfloor$ then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Proof If π is a permutation of $[n]$ and $A \subseteq [n]$ let

$$\theta(\pi, A) = \begin{cases} 1 & \exists s: \ \{\pi(s), \pi(s+1), \ldots, \pi(s+k-1)\} = A \\ 0 & otherwise \end{cases}$$

where $\pi(i) = \pi(i-n)$ if $i > n$.

We will show that for any permutation π,

$$\sum_{A \in \mathcal{A}} \theta(\pi, A) \leq k. \tag{9}$$

Assume (9). We first observe that if π is a random permutation then

$$\mathbf{E}(\theta(\pi, A)) = n\frac{k!(n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

and so, from (9),

$$k \geq \mathbf{E}(\sum_{A\in\mathcal{A}} \theta(\pi, A)) = \sum_{A\in\mathcal{A}} \frac{k}{\binom{n-1}{|A|-1}}$$

Hence

$$|\mathcal{A}| \leq \binom{n-1}{k-1}$$

Assume w.l.o.g. that π is the identity permutation.

Let $A_t = \{t, t+1, \dots, t+k-1\}$ and suppose that $A_s \in \mathcal{A}$.

All of the other sets A_t that intersect A_s can be partitioned into pairs A_{s-i}, A_{s+k-i}, $1 \leq i \leq k-1$ and the members of each pair are disjoint. Thus $\mathcal{A}$ can contain at most one from each pair. This verifies (9).

Kraft's Inequality

Let $x_1, x_2, \ldots, x_m$ be a collection of sequences over an alphabet Σ of size s. Let x_i have length n_i and let $n = \max\{n_1, n_2, \ldots, n_m\}$.

Assume next that no sequence is a prefix of any other sequence: Sequence $x_i = a_1 a_2 \cdots a_{n_i}$ is a prefix of $x_j = b_1 b_2 \cdots b_{n_j}$ if $a_i = b_i$ for $i = 1, 2, \ldots, n_i$.

Theorem

$$\sum_{i=1}^{m} r^{-n_i} \leq 1.$$

Proof: Let x be a random sequence of length n. Let $\mathcal{E}_i$ be the event x_i is a prefix of x. Then

(a) $\mathbf{Pr}(\mathcal{E}_i) = r^{-n_i}$.

(b) The event $\mathcal{E}_i$, $i = 1, 2, \ldots, m$ are disjoint.
(If $\mathcal{E}_i$ and $\mathcal{E}_j$ both occur and $n_i \leq n_j$ then x_i is a prefix of x_j.

Property (b) implies that

$$\mathbf{Pr}\left(\bigcup_{i=1}^{m} \mathcal{E}_i\right) = \mathbf{Pr}(\mathcal{E}_1) + \mathbf{Pr}(\mathcal{E}_2) + \cdots + \mathbf{Pr}(\mathcal{E}_m) \leq 1.$$

The theorem now follows from Property (a). □

Sunflowers

A sunflower of size r is a family of sets $A_1, A_2, \dots, A_r$ such that every element that belongs to more than one of the sets belongs to all of them.

Let $f(k, r)$ be the maximum size of a family of k-sets without a sunflower of size r.

Theorem

$f(k, r) \leq (r-1)^k k!$.

Proof Let $\mathcal{F}$ be a family of k-sets without a sunflower of size r. Let $A_1, A_2, \dots, A_t$ be a maximum subfamily of pairwise disjoint subsets in $\mathcal{F}$.

Since a family of pairwise disjoint is a sunflower, we must have $t < r$.

Now let $A = \bigcup_{i=1}^{t} A_i$. For every $a \in A$ consider the family $\mathcal{F}_a = \{S \setminus \{a\} : S \in \mathcal{F}, a \in S\}$.

Now the size of A is at most $(r-1)k$.

The size of each $\mathcal{F}_a$ is at most $f(k-1, r)$. This is because a sunflower in $\mathcal{F}_a$ is a sunflower in $\mathcal{F}$.

So,

$$f(k,r) \leq (r-1)k \times f(k-1,r) \leq (r-1)k \times (r-1)^{k-1}(k-1)!,$$

by induction. □

10/1/2021

Odd Town In order to cut down the number of committees a town of n people has instituted the following rules:

(a) Each club shall have an odd number of members.
(b) Each pair of clubs shall share an even number of members.

Theorem

With these rules, there are at most n clubs.

Proof Suppose that the clubs are $C_1, C_2, \ldots, C_m \subseteq [n]$.

Let $\bar{\nu}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$ denote the incidence vector of C_i for $1 \leq i \leq m$ i.e. $v_{i,j} = 1$ iff $j \in C_i$. We treat these vectors as being over the two element field $\mathbb{F}_2$.

We claim that $\bar{\nu}_1, \bar{\nu}_2, \ldots, \bar{\nu}_m$ are linearly independent and the theorem will follow.

The rules imply that (i) $\bar{\nu}_i \cdot \bar{\nu}_i = 1$ and (ii) $\bar{\nu}_i \cdot \bar{\nu}_j = 0$ for $1 \leq i \neq j \leq m$.
(Remember that we are working over $\mathbb{F}_2$.)

Suppose then that

$$c_1\bar{\nu}_1 + c_2\bar{\nu}_2 + \cdots + c_m\bar{\nu}_m = 0.$$

We show that $c_1 = c_2 = \cdots = c_m = 0$.

Indeed, we have

$$\begin{aligned} 0 &= \bar{\nu}_j \cdot (c_1\bar{\nu}_1 + c_2\bar{\nu}_2 + \cdots + c_m\bar{\nu}_m) \\ &= c_1\bar{\nu}_1 \cdot \bar{\nu}_j + c_2\bar{\nu}_2 \cdot \bar{\nu}_j + \cdots + c_m\bar{\nu}_m \cdot \bar{\nu}_j \\ &= c_j, \end{aligned}$$

for $j = 1, 2, \ldots, m$. □

Decomposing K_n into bipartite subgraphs: here we show

Theorem

If $G_k, k = 1, 2, \ldots, m$ is a collection of complete bipartite graphs with vertex partitions A_k, B_k, such that every edge of K_n is in exactly one subgraph, then $m \geq n - 1$. (Note that $A_k \cap B_k = \emptyset$ here.)

Proof This is tight since we can take $A_k = \{k\}, B_k = \{k + 1, \ldots, n\}$ for $k = 1, 2, \ldots, n - 1$.

Define $n \times n$ matrices M_k where $M_k(i, j) = 1$ if $i \in A_k, j \in B_k$ and $M_k(i, j) = 0$ otherwise.

Let $S = M_1 + M_2 + \cdots + M_m$. Then $S + S^T = J_n - I_n$ where I_n is the identity matrix and J_n is the all ones matrix.

We show next that $rank(S) \geq n-1$ and then the theorem follows from

$$rank(S) \leq rank(M_1) + rank(M_2) + \cdots + rank(M_m) \leq m.$$

Suppose then that $rank(S) \leq n-2$ so that there exists a non-zero solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ to the system of equations

$$S\mathbf{x} = 0, \ \sum_{i=1}^{n} x_i = 0.$$

But then, $J_n\mathbf{x} = 0$ and $S^T\mathbf{x} = -\mathbf{x}$ and $-|\mathbf{x}|^2 = -\mathbf{x}^T S^T \mathbf{x} = 0$, contradiction. □

Nonuniform Fisher Inequality;

Theorem

Let $C_1, C_2, \ldots, C_m$ be distinct subsets of $[n]$ such that for every $i \neq j$ we have $|C_i \cap C_j| = s$ where $1 \leq s < n$. Then $m \leq n$.

Proof If $|C_1| = s$ then $C_i \supset C_1, i = 2, 3, \ldots, m$ and the sets $C_i \setminus C_1$ are pairwise disjoint for $i \geq 2$.

It follows in this case that $m \leq 1 + n - s \leq n$.

Assume from now on that $c_i = |C_i| - s > 0$ for $i \in [m]$.

Let M be the $m \times n$ $0/1$ matrix where $M(i,j) = 1$ iff $j \in C_i$.

Let

$$A = MM^T = sJ + D$$

where J is the $m \times m$ all 1's matrix and D is the diagonal matrix, where $D(i,i) = c_i$.

We show that A and hence M has rank m, implying that $m \leq n$ as claimed.

We will in fact show that $\mathbf{x}^T A\mathbf{x} > 0$ for all $0 \neq \mathbf{x} \in \Re^m$. This means that $A\mathbf{x} \neq 0$ when $\mathbf{x} \neq 0$.

If $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ then

$$\mathbf{x}^T A\mathbf{x} = s(x_1 + x_2 + \cdots + x_m)^2 + \sum_{i=1}^{m} c_i x_i^2 > 0.$$

$\square$

10/4/2021

Pigeon Hole Principle

We have two disks, each partitioned into 200 sectors of the same size. 100 of the sectors of Disk 1 are coloured Red and 100 are colored Blue. The 200 sectors of Disk 2 are arbitrarily coloured Red and Blue.

It is always possible to place Disk 2 on top of Disk 1 so that the centres coincide, the sectors line up and at least 100 sectors of Disk 2 have the same colour as the sector underneath them.

Fix the position of Disk 1. There are 200 positions for Disk 2 and let q_i denote the number of matches if Disk 2 is placed in position i. Now for each sector of Disk 2 there are 100 positions i in which the colour of the sector underneath it coincides with its own.

Therefore

$$q_1 + q_2 + \cdots + q_{200} = 200 \times 100 \qquad (10)$$

and so there is an i such that $q_i \geq 100$.

Explanation of (19).
Consider 0-1 200×200 matrix $A(i,j)$ where $A(i,j) = 1$ iff sector j lies on top of a sector with the same colour when in position i. Row i of A has q_i 1's and column j of A has 100 1's. The LHS of (19) counts the number of 1's by adding rows and the RHS counts the number of 1's by adding columns.

Pigeon Hole Principle

Alternative solution: Place Disk 2 randomly on Disk 1 so that the sectors align. For $i = 1, 2, \ldots, 200$ let

$$X_i = \begin{cases} 1 & \textit{sector } i \textit{ of disk 2 is on sector of disk 1 of same color} \\ 0 & \textit{otherwise} \end{cases}$$

We have

$$\mathbf{E}(X_i) = 1/2 \qquad \textit{for } i = 1, 2, \ldots, 200.$$

So if $X = X_1 + \cdots + X_{200}$ is the number of sectors sitting above sectors of the same color, then $\mathbf{E}(X) = 100$ and there must exist at least one way to achieve 100.

Theorem

(Erdős-Szekeres) An arbitrary sequence of integers $(a_1, a_2, \ldots, a_{k^2+1})$ *contains a monotone subsequence of length* $k+1$*.*

Proof. Let $(a_i, a_i^1, a_i^2, \ldots, a_i^{\ell-1})$ be the longest *monotone increasing* subsequence of $(a_1, \ldots, a_{k^2+1})$ that starts with $a_i, (1 \leq i \leq k^2+1)$, and let $\ell(a_i)$ be its length.

If for some $1 \leq i \leq k^2+1$, $\ell(a_i) \geq k+1$, then $(a_i, a_i^1, a_i^2, \ldots, a_i^{l-1})$ is a monotone increasing subsequence of length $\geq k+1$.

So assume that $\ell(a_i) \leq k$ holds for every $1 \leq i \leq k^2+1$.

Pigeon Hole Principle

Consider k holes $1, 2, \ldots, k$ and place i into hole $\ell(a_i)$.

There are $k^2 + 1$ subsequences and $\leq k$ non-empty holes (different lengths), so by the pigeon-hole principle there will exist ℓ^* such that there are (at least) $k + 1$ indices $i_1 < i_2 < \cdots < i_{k+1}$ such that $\ell(a_{i_t}) = \ell^*$ for $1 \leq t \leq k + 1$.

Then we must have $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$.

Indeed, assume to the contrary that $a_{i_m} < a_{i_n}$ for some $1 \leq m < n \leq k + 1$. Then $a_{i_m} \leq a_{i_n} \leq a^1_{i_n} \leq a^2_{i_n} \leq \cdots \leq a^{\ell^*-1}_{i_n}$, i.e., $\ell(a_{i_m}) \geq \ell^* + 1$, a contradiction. $\square$

The sequence

$$n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, \ldots, n^2, n^2-1, \ldots, n^2-n+1$$

has no monotone subsequence of length $n+1$ and so the Erdős-Szekerés result is best possible.

Pigeon Hole Principle

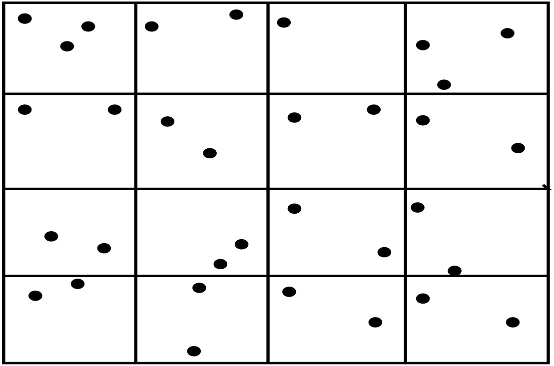
Let $P_1, P_2, \ldots, P_n$ be n points in the unit square $[0,1]^2$. We will show that there exist $i, j, k \in [n]$ such that the triangle $P_iP_jP_k$ has area

$$\leq \frac{1}{2(\left\lfloor\sqrt{(n-1)/2}\right\rfloor)^2} \sim \frac{1}{n}$$

for large n.

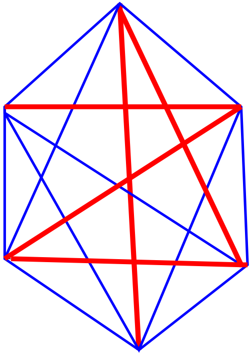
Pigeon Hole Principle

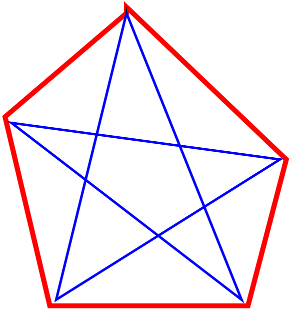
Let $m = \left\lfloor \sqrt{(n-1)/2} \right\rfloor$ and divide the square up into $m^2 < \frac{n}{2}$ subsquares. By the pigeonhole principle, there must be a square containing ≥ 3 points. Let 3 of these points be $P_i P_j P_k$. The area of the corresponding triangle is at most one half of the area of an individual square.



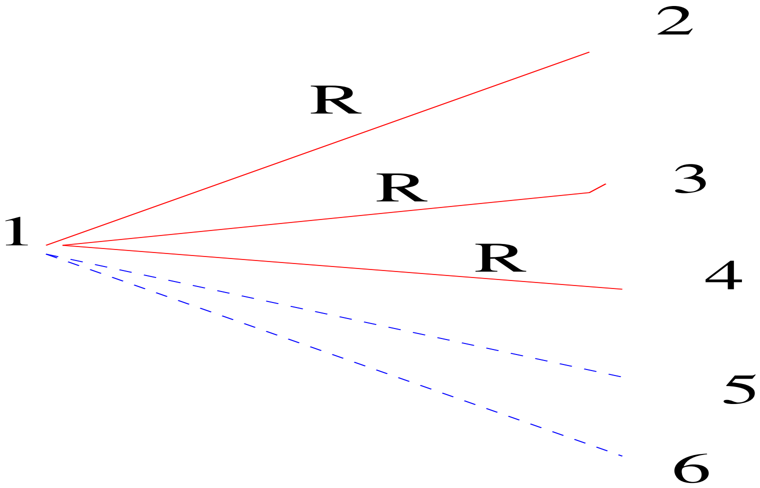
Ramsey Theory

Suppose we 2-colour the edges of K_6 of Red and Blue. There *must* be either a Red triangle or a Blue triangle.





This is not true for K_5.



There are 3 edges of the same colour incident with vertex 1, say (1,2), (1,3), (1,4) are Red. Either (2,3,4) is a blue triangle or one of the edges of (2,3,4) is Red, say (2,3). But the latter

Ramsey's Theorem

For all positive integers k, ℓ there exists $R(k, \ell)$ such that if $N \geq R(k, \ell)$ and the edges of K_N are coloured Red or Blue then then either there is a "Red k-clique" or there is a "Blue ℓ-clique. A clique is a complete subgraph and it is Red if all of its edges are coloured red etc.

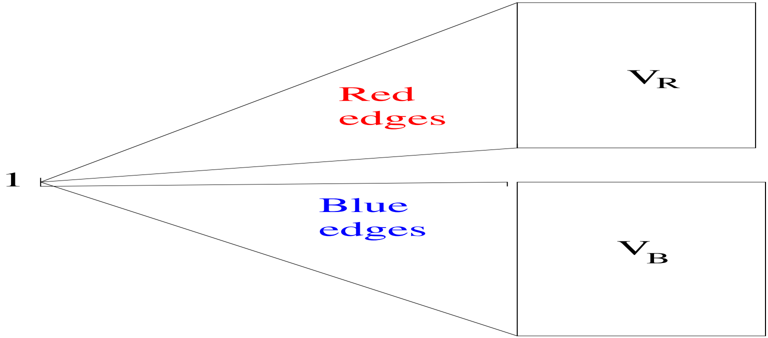
$$\begin{aligned} R(1,k) &= R(k,1) &= 1 \\ R(2,k) &= R(k,2) &= k \end{aligned}$$

10/6/2021

Theorem

$$R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell).$$

Proof Let $N = R(k,\ell-1) + R(k-1,\ell)$.



$V_R = \{(x : (1,x)$ is coloured Red$\}$ and $V_B = \{(x : (1,x)$ is

$$|V_R| \geq R(k-1,\ell) \text{ or } |V_B| \geq R(k,\ell-1).$$

Since

$$\begin{aligned} |V_R| + |V_B| &= N - 1 \\ &= R(k,\ell-1) + R(k-1,\ell) - 1. \end{aligned}$$

Suppose for example that $|V_R| \geq R(k-1,\ell)$. Then either V_R contains a Blue ℓ-clique – done, or it contains a Red $k-1$-clique K. But then $K \cup \{1\}$ is a Red k-clique. Similarly, if $|V_B| \geq R(k,\ell-1)$ then either V_B contains a Red k-clique – done, or it contains a Blue $\ell-1$-clique L and then $L \cup \{1\}$ is a Blue ℓ-clique. □

Theorem

$$R(k,\ell) \le \binom{k+\ell-2}{k-1}.$$

Proof Induction on $k+\ell$. True for $k+\ell \le 5$ say. Then

$$\begin{aligned} R(k,\ell) &\le R(k,\ell-1)+R(k-1,\ell) \\ &\le \binom{k+\ell-3}{k-1}+\binom{k+\ell-3}{k-2} \\ &= \binom{k+\ell-2}{k-1}. \end{aligned}$$

□

So, for example,

$$R(k,k) \le \binom{2k-2}{k-1}$$

Theorem

$$R(k,k) > 2^{k/2}$$

Proof We must prove that if $n \leq 2^{k/2}$ then there exists a Red-Blue colouring of the edges of K_n which contains no Red k-clique and no Blue k-clique. We can assume $k \geq 4$ since we know $R(3,3) = 6$.
We show that this is true with positive probability in a *random* Red-Blue colouring. So let Ω be the set of all Red-Blue edge colourings of K_n with uniform distribution. Equivalently we independently colour each edge Red with probability 1/2 and Blue with probability 1/2.

Let
$\mathcal{E}_R$ be the event: {There is a Red k-clique} and
$\mathcal{E}_B$ be the event: {There is a Blue k-clique}.
We show

$$\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) < 1.$$

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be the vertices of the N k-cliques of K_n.

Let $\mathcal{E}_{R,j}$ be the event: {C_j is Red} and let $\mathcal{E}_{B,j}$ be the event: {C_j is Blue}.

$$
\begin{aligned}
\mathbf{Pr}(\mathcal{E}_R \cup \mathcal{E}_B) &\leq \mathbf{Pr}(\mathcal{E}_R) + \mathbf{Pr}(\mathcal{E}_B) = 2\mathbf{Pr}(\mathcal{E}_R) \\
&= 2\mathbf{Pr}\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) \leq 2\sum_{j=1}^{N} \mathbf{Pr}(\mathcal{E}_{R,j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \frac{2^{1+k/2}}{k!} \\
&< 1.
\end{aligned}
$$

Ramsey Theory

Very few of the Ramsey numbers are known exactly. Here are a few known values.

$$\begin{aligned} R(3,3) &= 6 \\ R(3,4) &= 9 \\ R(4,4) &= 18 \\ R(4,5) &= 25 \\ 43 \leq R(5,5) &\leq 49 \end{aligned}$$

Schur's Theorem

Let $r_k = N(3, 3, \ldots, 3; 2)$ be the smallest n such that if we k-color the edges of K_n then there is a mono-chromatic triangle.

Theorem

For all partitions $S_1, S_2, \ldots, S_k$ of $[r_k]$, there exist i and $x, y, z \in S_i$ such that $x + y = z$.

Proof Given a partition $S_1, S_2, \ldots, S_k$ of $[n]$ where $n \geq r_k$ we define a coloring of the edges of K_n by coloring (u, v) with color j where $|u - v| \in S_j$.

There will be a mono-chromatic triangle i.e. there exist j and $x < y < z$ such that $u = y - x$, $v = z - x$, $w = z - y \in S_j$. But $u + v = w$. □

A set of points X in the plane is in general position if no 3 points of X are collinear.

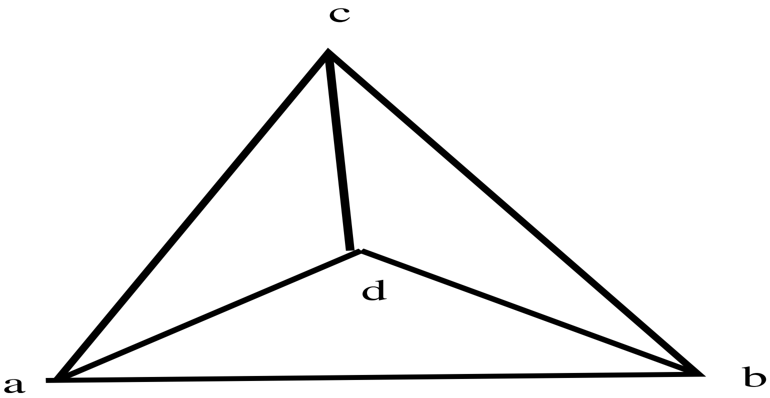
Theorem

If $n \geq N(k,k;3)$ and X is a set of n points in the plane which are in general position then X contains a k-subset Y which form the vertices of a convex polygon.

Proof We first observe that if every 4-subset of $Y \subseteq X$ forms a convex quadrilateral then Y itself induces a convex polygon.

Now label the points in S from X_1 to X_n and then color each triangle $T = \{X_i, X_j, X_k\},\ i < j < k$ as follows: If traversing triangle $X_iX_jX_k$ in this order goes round it clockwise, color T Red, otherwise color T Blue.

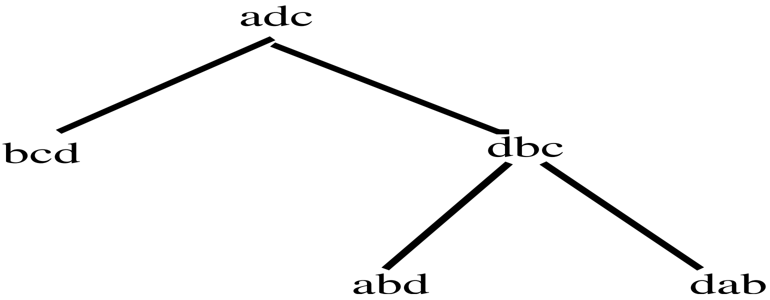
Now there must exist a k-set T such that all triangles formed from T have the same color. All we have to show is that T does not contain the following configuration:



Assume w.l.o.g. that $a < b < c$ which implies that $X_i X_j X_k$ is colored Blue.

All triangles in the previous picture are colored Blue.

So the possibilities are



and all are impossible.

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We define $r(H_1, H_2)$ to be the minimum n such that in in Red-Blue coloring of the edges of K_n there is eithere (i) a Red copy of H_1 or (ii) a Blue copy of H_2.

As an example, consider $r(P_3, P_3)$ where P_t denotes a path with t edges.

We show that

$$r(P_3, P_3) = 5.$$

$R(P_3, P_3) > 4$: We color edges incident with 1 Red and the remaining edges $\{(2,3),(3,4),(4,1)\}$ Blue. There is no mono-chromatic P_3.

$R(P_3, P_3) \leq 5$: There must be two edges of the same color incident with 1.

Assume then that $(1, 2), (1, 3)$ are both Red.

If any of $(2, 4), (2, 5), (3, 4), (3, 5)$ are Red then we have a Red P_3.

If all four of these edges are Blue then $(4, 2, 5, 3)$ is Blue.

We show next that $r(K_{1,s}, P_t) \leq s + t$. Here $K_{1,s}$ is a star: i.e. a vertex v and t incident edges.

Let $n = s + t$. If there is no vertex of Red degree s then the minimum degree in the graph induced by the Blue edges is at least t.

We then note that a graph of minimum degree δ contains a path of length δ.

Partially Ordered Sets

A **partially ordered set** or **poset** is a set P and a binary relation $\preceq$ such that for all $a, b, c \in P$

1. $a \preceq a$ (reflexivity).
2. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).
3. $a \preceq b$ and $b \preceq a$ implies $a = b$. (anti-symmetry).

Examples

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \preceq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\preceq = \subseteq$.

A pair of elements a, b are **comparable** if $a \preceq b$ or $b \preceq a$. Otherwise they are **incomparable**.

A poset without incomparable elements (Example 1) is a linear or total order.

We write $a < b$ if $a \preceq b$ and $a \neq b$.

A **chain** is a sequence $a_1 < a_2 < \cdots < a_s$.

A set A is an **anti-chain** if every pair of elements in A are incomparable.

Thus a Sperner family is an anti-chain in our third example.

Theorem

Let P be a finite poset, then
$\min\{m : \exists$ *anti-chains* $A_1, A_2, \ldots, A_\mu$ *with* $P = \bigcup_{i=1}^{\mu} A_i\}$ =
$\max\{|C| : A$ *is a chain*$\}$.

The minimum number of anti-chains needed to cover P is at least the size of any chain, since a chain can contain at most one element from each anti-chain.

Partially Ordered Sets

We prove the converse by induction on the maximum length μ of a chain. We have to show that P can be partitioned into μ anti-chains.

If $\mu = 1$ then P itself is an anti-chain and this provides the basis of the induction.

So now suppose that $C = x_1 < x_2 < \cdots < x_\mu$ is a maximum length chain and let A be the set of maximal elements of P.

(An element is x *maximal* if $\nexists y$ such that $y > x$.)

A is an anti-chain.

Now consider $P' = P \setminus A$. P' contains no chain of length μ. If it contained $y_1 < y_2 < \cdots < y_\mu$ then since $y_\mu \notin A$, there exists $a \in A$ such that P contains the chain $y_1 < y_2 < \cdots < y_\mu < a$, contradiction.

Thus the maximum length of a chain in P' is $\mu - 1$ and so it can be partitioned into anti-chains $A_1 \cup A_2 \cup \cdots A_{\mu-1}$. Putting $A_\mu = A$ completes the proof. □

Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that A is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of A in some chain.

Theorem

(Dilworth) Let P be a finite poset, then
$\min\{m : \exists \text{ chains } C_1, C_2, \ldots, C_m \text{ with } P = \bigcup_{i=1}^{m} C_i\} =$
$\max\{|A| : A \text{ is an anti-chain}\}$.

Intervals Problem

$I_1, I_2, \ldots, I_{mn+1}$ are closed intervals on the real line i.e. $I_j = [a_j, b_j]$ where $a_j \leq b_j$ for $1 \leq j \leq mn+1$.

Theorem

Either (i) there are $m+1$ intervals that are pair-wise disjoint or (ii) there are $n+1$ intervals with a non-empty intersection

Define a partial ordering $\preceq$ on the intervals by $I_r \preceq I_s$ iff $b_r \leq a_s$. Suppose that $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ is a collection of pair-wise disjoint intervals. Assume that $a_{i_1} < a_{i_2} \cdots < a_{i_t}$. Then $I_{i_1} < I_{i_2} \cdots < I_{i_t}$ form a chain and conversely a chain of length t comes from a set of t pair-wise disjoint intervals.
So if (i) does not hold, then the maximum length of a chain is m.

This means that the minimum number of chains needed to cover the poset is at least $\lceil \frac{mn+1}{m} \rceil = n + 1$.

Dilworth's theorem implies that there must exist an anti-chain $\{I_{j_1}, I_{j_2}, \ldots, I_{j_{n+1}}\}$.

Let $a = \max\{a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}}\}$ and $b = \min\{b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}}\}$.

We must have $a \leq b$ else the two intervals giving a, b are disjoint.

But then every interval of the anti-chain contains $[a, b]$.

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Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that A is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of A in some chain.

Theorem

(Dilworth) Let P be a finite poset, then $\min\{m : \exists \text{ chains } C_1, C_2, \ldots, C_m \text{ with } P = \bigcup_{i=1}^{m} C_i\} = \max\{|A| : A \text{ is an anti-chain}\}$.

We have already argued that $\max\{|A|\} \leq \min\{m\}$.

We will prove there is equality here by induction on $|P|$.

The result is trivial if $|P| = 0$.

Now assume that $|P| > 0$ and that μ is the maximum size of an anti-chain in P. We show that P can be partitioned into μ chains.

Let $C = x_1 < x_2 < \cdots < x_p$ be a *maximal* chain in P i.e. we cannot add elements to it and keep it a chain.

Case 1 Every anti-chain in $P \setminus C$ has $\leq \mu - 1$ elements. Then by induction $P \setminus C = \bigcup_{i=1}^{\mu-1} C_i$ and then $P = C \cup \bigcup_{i=1}^{\mu-1} C_i$ and we are done.

Case 2

There exists an anti-chain $A = \{a_1, a_2, \ldots, a_\mu\}$ in $P \setminus C$. Let

- $P^- = \{x \in P: \; x \preceq a_i$ for some $i\}$.
- $P^+ = \{x \in P: \; x \succeq a_i$ for some $i\}$.

Note that

1. $P = P^- \cup P^+$. Otherwise there is an element x of P which is incomparable with every element of A and so μ is not the maximum size of an anti-chain.
2. $P^- \cap P^+ = A$. Otherwise there exists x, i, j such that $a_i < x < a_j$ and so A is not an anti-chain.
3. $x_p \notin P^-$. Otherwise $x_p < a_i$ for some i and the chain C is not maximal.

Applying the inductive hypothesis to P^- ($|P^-| < |P|$ follows from 3) we see that P^- can be partitioned into μ chains $C_1^-, C_2^-, \ldots, C_\mu^-$.

Now the elements of A must be distributed one to a chain and so we can assume that $a_i \in C_i^-$ for $i = 1, 2, \ldots, \mu$.

a_i must be the maximum element of chain C_i^-, else the maximum of C_i^- is in $(P^- \cap P^+) \setminus A$, which contradicts 2.

Applying the same argument to P^+ we get chains $C_1^+, C_2^+, \ldots, C_\mu^+$ with a_i as the minimum element of C_i^+ for $i = 1, 2, \ldots, \mu$.

Then from 2 we see that $P = C_1 \cup C_2 \cup \cdots \cup C_\mu$ where $C_i = C_i^- \cup C_i^+$ is a chain for $i = 1, 2, \ldots, \mu$. □

Three applications of Dilworth's Theorem

(i) Another proof of

Theorem

Erdős and Szekerés

$a_1, a_2, \ldots, a_{n^2+1}$ *contains a monotone subsequence of length* $n+1$.

Let $P = \{(i, a_i) : 1 \le i \le n^2 + 1\}$ and let say $(i, a_i) \preceq (j, a_j)$ if $i < j$ and $a_i \le a_j$.

A chain in P corresponds to a monotone increasing subsequence. So, suppose that there are no monotone increasing sequences of length $n+1$. Then any cover of P by chains requires at least $n+1$ chains and so, by Dilworths theorem, there exists an anti-chain A of size $n+1$.

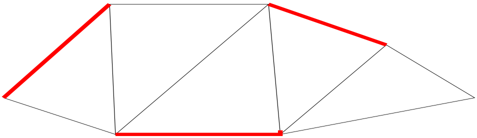
Let $A = \{(i_t, a_{i_t}) : 1 \leq t \leq n+1\}$ where $i_1 < i_2 \leq \cdots < i_{n+1}$.

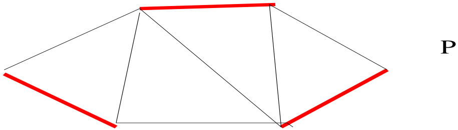
Observe that $a_{i_t} > a_{i_{t+1}}$ for $1 \leq t \leq n$, for otherwise $(i_t, a_{i_t}) \preceq (i_{t+1}, a_{i_{t+1}})$ and A is not an anti-chain.

Thus A defines a monotone decreasing sequence of length $n+1$. □

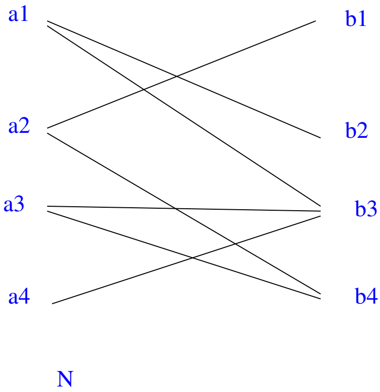
Matchings in bipartite graphs

Re-call that a matching is a set of vertex disjoint edges.





Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.
For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.

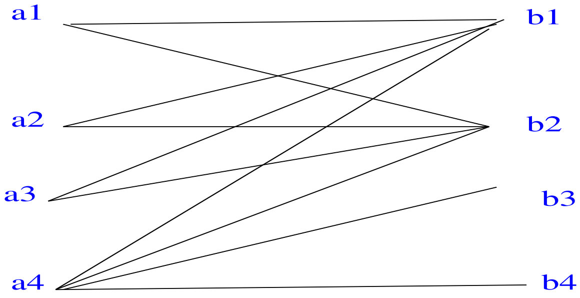


Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Theorem

(Hall) G contains a matching of size $|A|$ iff

$$|N(S)| \geq |S| \qquad \forall S \subseteq A.$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

If G contains a matching M of size $|A|$ then $M = \{(a, f(a)) : \ a \in A\}$, where $f : A \to B$ is a 1-1 function.

But then,

$$|N(S)| \geq |f(S)| = S$$

for all $S \subseteq A$.

Let $G = (A \cup B, E)$ be a bipartite graph which satisfies Hall's condition. Define a poset $P = A \cup B$ and define $<$ by $a < b$ only if $a \in A, b \in B$ and $(a, b) \in E$.

Suppose that the largest anti-chain in P is $A = \{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k\}$ and let $s = h + k$.

Now

$$N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}$$

for otherwise A will not be an anti-chain.

From Hall's condition we see that

$$|B| - k \geq h \text{ or equivalently} |B| \geq s.$$

Now by Dilworth's theorem, P is the union of s chains:

A matching M of size m, $|A| - m$ members of A and $|B| - m$ members of B.

But then

$$m + (|A| - m) + (|B| - m) = s \leq |B|$$

and so $m \geq |A|$. $\square$

10/13/2021

A *network* consists of a loopless digraph $D = (V, A)$ plus a function $c : A \to \boldsymbol{R}_+$. Here $c(x, y)$ for $(x, y) \in A$ is the *capacity* of the edge (x, y).

We use the following notation: if $\phi : A \to \boldsymbol{R}$ and S, T are (not necessarily disjoint) subsets of V then

$$\phi(S, T) = \sum_{\substack{x \in S \\ y \in T}} \phi(x, y).$$

Let s, t be distinct vertices. An $s - t$ flow is a function $f : A \to \boldsymbol{R}$ such that

$$f(v, V \setminus \{v\}) = f(V \setminus \{v\}, v) \quad \text{for all } v \neq s, t.$$

In words: flow into v equals flow out of v.

An $s-t$ flow is *feasible* if

$$0 \le f(x,y) \le c(x,y) \quad \text{for all } (x,y) \in A.$$

An $s-t$ *cut* is a partition of V into two sets $S, \bar{S}$ such that $s \in S$ and $t \in \bar{S}$.

The *value* v_f of the flow f is given by

$$v_f = f(s, V \setminus \{s\}) - f(V \setminus \{s\}, s).$$

Thus v_f is the net flow leaving s.

The *capacity* of the cut $S:\bar{S}$ is equal to $c(S,\bar{S})$.

Max-Flow Min-Cut Theorem

Theorem

$$\max v_f = \min c(S, \bar{S})$$

where the maximum is over feasible $s - t$ *flows and the minimum is over* $s - t$ *cuts.*

Proof We observe first that

$$\begin{aligned} f(S, \bar{S}) - f(\bar{S}, S) &= (f(S, V) - f(S, S)) - (f(V, S) - f(S, S)) \\ &= f(S, V) - f(V, S) \\ &= v_f + \sum_{v \in S \setminus \{s\}} (f(v, V) - f(V, v)) \\ &= v_f. \end{aligned}$$

So,

$$v_f \leq f(S, \bar{S}) \leq c(S, \bar{S}).$$

This implies that

$$\max v_f \leq \min c(S, \bar{S}). \tag{11}$$

Given a flow f we define a *flow augmenting path* P to be a sequence of distinct vertices $x_0 = s, x_1, x_2, \ldots, x_k = t$ such that for all i, either

- F1 $(x_i, x_{i+1}) \in A$ and $f(x_i, x_{i+1}) < c(x_i, x_{i+1})$, or
- F2 $(x_{i+1}, x_i) \in A$ and $f(x_{i+1}, x_i) > 0$.

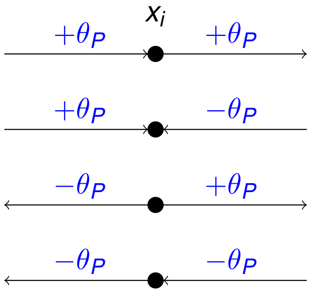
If P is such a sequence, then we define $\theta_P > 0$ to be the minimum over i of $c(x_i, x_{i+1}) - f(x_i, x_{i+1})$ (Case (F1)) and $f(x_{i+1}, x_i)$ (Case (F2)).

Claim 1: f is a maximum value flow, iff there are no flow augmenting paths.

Proof If P is flow augmenting then define a new flow f' as follows:

1. $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \theta_P$ or
2. $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \theta_P$
3. For all other edges, (x, y), we have $f'(x, y) = f(x, y)$.

We can see that the flow stays balanced at x_i.



We can see then that if there is a flow augmenting path then the new flow satisfies

$$v_{f'} = v_f + \theta_P > v_f.$$

Let S_f denote the set of vertices v for which there is a sequence $x_0 = s, x_1, x_2, \ldots, x_k = v$ which satisfies F1, F2 of the definition of flow augmenting paths.

If $t \in S_f$ then the associated sequence defines a flow augmenting path. So, assume that $t \notin S_f$. Then we have,

1. $s \in S_f$.
2. If $x \in S_f, y \in \bar{S}_f, (x, y) \in A$ then $f(x, y) = c(x, y)$, else we would have $y \in S_f$.
3. If $x \in S_f, y \in \bar{S}_f, (y, x) \in A$ then $f(y, x) = 0$, else we would have $y \in S_f$.

We therefore have

$$
\begin{aligned}
v_f &= f(S_f, \bar{S}_f) - f(\bar{S}_f, S) \\
&= c(S, \bar{S}_f).
\end{aligned}
$$

We see from this and (11) that f is a flow of maximum value and that the cut $S_f : \bar{S}_f$ is of minimum capacity.

This finishes the proof of Claim 1 and the Max-Flow Min-Cut theorem.

Note also that we can construct S_f by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds. (A simple inductive argument based on sequence length shows that all of S_f is constructed in this way.)

Note also that we can construct S_f by beginning with $S_f = \{s\}$ and then repeatedly adding any vertex $y \notin S_f$ for which there is $x \in S_f$ such that F1 or F2 holds.

This defines an algorithm for finding a maximum flow. The construction either finishes with $t \in S_f$ and we can augment the flow.

Or, we find that $t \notin S_f$ and we have a maximum flow.

Note, that if all the capacities $c(x, y)$ are integers and we start with the all zero flow then we find that θ_f is always a positive integer (formally one can use induction to verify this).

It follows that in this case, there is always a maximum flow that only takes integer values on the edges.

Hall's Theorem.

Let $G = (A, B, E)$ be a bipartite graph with $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. A matching M is a set of edges that meets each vertex at most once. A matching is perfect if it meets each vertex.

Hall's theorem:

Theorem

G contains a perfect matching iff $|N(S)| \geq |S|$ for all $S \subseteq A$.

Here $N(S) = \{b \in B : \exists a \in A \text{ s.t. } \{a, b\} \in E\}$.

Define a digraph Γ by adding vertices $s, t \notin A \cup B$. Then add edges (s, a_i) and (b_i, t) of capacity 1 for $i = 1, 2, \dots, n$. Orient the edges E for A to B and give them capacity ∞.

G has a matching of size m iff there is an $s-t$ flow of value m. An $s-t$ cut $X:\bar{X}$ has capacity

$$|A\setminus X|+|B\cap X|+|\{a\in X\cap A, b\in B\setminus X:\{a,b\}\in E\}|\times\infty.$$

It follows that to find a minimum cut, we need only consider X such that

$$\{a\in X\cap A, b\in B\setminus X:\{a,b\}\in E\}=\emptyset. \tag{12}$$

For such a set, we let $S=A\cap X$ and $T=X\cap B$. Condition (12) means that $T\supseteq N(S)$. The capacity of $X:\bar{X}$ is now $(n-|S|)+|T|$ and for a fixed S this is minimised for $T=N(S)$.

Thus, by the Max-Flow Min-Cut theorem

$$\max\{|M|\}=\min_X\{c(X:\bar{X})\}=\min_S\{n-|S|+|N(S)|\}.$$

This implies Hall's theorem.

10/15/2021

Graph orientation problem

Let $G = (V, E)$ be a graph. When is it possible to orient the edges of G to create a digraph $\Gamma = (V, A)$ so that every vertex has out-degree at least d. We say that G is d-orientable.

Theorem

G is d-orientable iff

$$|\{e \in E : e \cap S \neq \emptyset\}| \geq d|S| \text{ for all } S \subseteq V. \tag{13}$$

Proof If G is d-orientable then

$$|\{e \in E : e \cap S \neq \emptyset\}| \geq |\{(x, y) \in A : x \in S\}| \geq d|S|.$$

Suppose now that (13) holds. Define a network D as follows; the vertices are s, t, V, E – yes, D has a vertex for each edge of G.

There is an edge of capacity d from s to each $v \in V$ and an edge of capacity one from each $e \in E$ to t. There is an edge of infinite capacity from $v \in V$ to each edge e that contains v.

Consider an integer flow f. Suppose that $e = \{v, w\} \in E$ and $f(e, t) = 1$. Then either $f(v, e) = 1$ or $f(w, e) = 1$. In the former we interpret this as orienting the edge e from v to w and in the latter from w to v.

Under this interpretation, G is d-orientable iff D has a flow of value $d|V|$.

Let $X : \bar{X}$ be an $s - t$ cut in N. Let $S = X \cap V$ and $T = X \cap E$.

To have a finite capacity, there must be no $x \in S$ and $e \in E \setminus T$ such that $x \in e$.

So, the capacity of a finite capacity cut is at least

$$d(|V| - |S|) + |\{e \in E : e \cap S \neq \emptyset\}|$$

And this is at least $d|V|$ if (13) holds.

Example 1 A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into n sectors of angle $2\pi/n$. Each sector is to be colored Red or Blue. How many different colorings are there?

One could argue for 2^n.

On the other hand, what if we only distinguish colorings which cannot be obtained from one another by a rotation. For example if $n = 4$ and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.

Example 2

Now consider an $n \times n$ "chessboard" where $n \geq 2$. Here we color the squares Red and Blue and two colorings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colorings.

The general scenario that we consider is as follows: We have a set X which will stand for the set of colorings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$).

In addition there is a set G of permutations of X. This set will have a group structure:

Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that G is *closed* under composiiton i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$.

We also have the following:

- A1 The *identity* permutation $1_X \in G$.
- A2 $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).
- A3 The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set G with a binary relation $\circ$ which satisfies **A1,A2,A3** is called a **Group**).

In example 1 $D = \{0, 1, 2, \ldots, n-1\}$, $X = 2^D$ and the group is $G_1 = \{e_0, e_1, \ldots, e_{n-1}\}$ where $e_j * x = x + j \; mod \; n$ stands for rotation by $2j\pi/n$.

In example 2, $X = 2^{[n]^2}$. We number the squares 1,2,3,4 in clockwise order starting at the upper left and represent X as a sequence from $\{r, b\}^4$ where for example rrbr means color 1,2,4 Red and 3 Blue. $G_2 = \{e, a, b, c, p, q, r, s\}$ is in a sense independent of n. e, a, b, c represent a rotation through $0, 90, 180, 270$ degrees respectively. p, q represent reflections in the vertical and horizontal and r, s represent reflections in the diagonals 1,3 and 2,4 respectively.

	e	a	b	c	p	q	r	s
rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr
brrr	brrr	rbrr	rrbr	rrrb	rbrr	rrrb	brrr	rrbr
rbrr	rbrr	rrbr	rrrb	brrr	brrr	rrbr	rrrb	rbrr
rrbr	rrbr	rrrb	brrr	rbrr	rrrb	rbrr	rrbr	brrr
rrrb	rrrb	brrr	rbrr	rrbr	rrbr	brrr	rbrr	rrrb
bbrr	bbrr	rbbr	rrbb	brrb	bbrr	rrbb	brrb	rbbr
rbbr	rbbr	rrbb	brrb	bbrr	brrb	rbbr	rrbb	bbrr
rrbb	rrbb	brrb	bbrr	rbbr	rrbb	bbrr	rbbr	brrb
brrb	brrb	bbrr	rbbr	rrbb	rbbr	brrb	bbrr	rrbb
rbrb	rbrb	brbr	rbrb	brbr	brbr	brbr	rbrb	rbrb
brbr	brbr	rbrb	brbr	rbrb	rbrb	rbrb	brbr	brbr
bbbr	bbbr	rbbb	brbb	bbrb	bbrb	rbbb	brbb	bbbr
bbrb	bbrb	bbbr	rbbb	brbb	bbbr	brbb	bbrb	rbbb
brbb	brbb	bbrb	bbbr	rbbb	brbb	bbrb	bbbr	brbb
rbbb	rbbb	brbb	bbrb	bbbr	brbb	bbbr	rbbb	bbrb
bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb

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Theorem 2

(Frobenius, Burnside)

$$\nu_{X,G} = \frac{1}{|G|}\sum_{g\in G} |Fix(g)|.$$

Proof Let $A(x,g) = 1_{g*x=x}$. Then

$$\begin{aligned}
\nu_{X,G} &= \frac{1}{|G|}\sum_{x\in X} |S_x| \\
&= \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G} A(x,g) \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X} A(x,g) \\
&= \frac{1}{|G|}\sum_{g\in G} |Fix(g)|.
\end{aligned}$$

Let us consider example 1 with $n = 6$. We compute

g	e_0	e_1	e_2	e_3	e_4	e_5
$\|Fix(g)\|$	64	2	4	8	4	2

Applying Theorem 2 we obtain

$$\nu_{X,G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14.$$

Cycles of a permutation

Let $\pi : D \to D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi = (D, A)$ where $A = \{(i, \pi(i)) : \ i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.
Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are $(1, 6, 8), (2), (3, 7, 9, 5), (4, 10)$.

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$.

Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i) \neq y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation.

So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i), i)$.

If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Example 2

It is straightforward to check that when n is even, we have

g	e	a	b	c	p	q	r	s
$\lvert Fix(g)\rvert$	2^{n^2}	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/4}$	$2^{n^2/2}$	$2^{n^2/2}$	$2^{n(n+1)/2}$	$2^{n(n+1)/2}$

For example, if we divide the chessboard into 4 $n/2 \times n/2$ sub-squares, numbered 1,2,3,4 then a coloring is in $Fix(a)$ iff each of these 4 sub-squares have colorings which are rotations of the coloring in square 1.

Polya's Theorem

We now extend the above analysis to answer questions like:
How many *distinct* ways are there to color an 8×8 chessboard with 32 white squares and 32 black squares?
The scenario now consists of a set D (*Domain*, a set C (colors) and $X = \{x : D \to C\}$ is the set of colorings of D with the color set C. G is now a group of permutations of D.

We see first how to extend each permutation of D to a permutation of X. Suppose that $x \in X$ and $g \in G$ then we define $g * x$ by

$$g * x(d) = x(g^{-1}(d)) \qquad \text{for all } d \in D.$$

Explanation: The color of d is the color of the element $g^{-1}(d)$ which is mapped to it by g.
Consider Example 1 with $n = 4$. Suppose that $g = e_1$ i.e. rotate clockwise by $\pi/2$ and $x(1) = b, x(2) = b, x(3) = r, x(4) = r$.
Then for example

$$g * x(1) = x(g^{-1}(1)) = x(4) = r, \text{ as before.}$$

Now associate a **weight** w_c with each $c \in C$.
If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(d)}.$$

Thus, if in Example 1 we let $w(r) = R$ and $w(b) = B$ and take $x(1) = b, x(2) = b, x(3) = r, x(4) = r$ then we will write $W(x) = B^2 R^2$.

For $S \subseteq X$ we define the **inventory** of S to be

$$W(S) = \sum_{x \in S} W(x).$$

The problem we discuss now is to compute the **pattern inventory** $PI = W(S^*)$ where S^* contains one member of each orbit of X under G.

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For example, in the case of Example 2, with $n = 2$, we get

$$PI = R^4 + R^3B + 2R^2B^2 + RB^3 + B^4.$$

To see that the definition of PI makes sense we need to prove

Lemma 3 If x, y are in the same orbit of X then $W(x) = W(y)$.

Proof Suppose that $g * x = y$. Then

$$\begin{aligned} W(y) &= \prod_{d \in D} w_{y(d)} \\ &= \prod_{d \in D} w_{g*x(d)} \\ &= \prod_{d \in D} w_{x(g^{-1}(d))} && (14) \\ &= \prod_{d \in D} w_{x(d))} && (15) \\ &= W(x) \end{aligned}$$

Note, that we can go from (14) to (15) because as d runs over D, $g^{-1}(d)$ also runs over d. □

Let $\Delta = |D|$. If $g \in G$ has k_i cycles of length i then we define

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$

The **Cycle Index Polynomial** of G, C_G is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2 x_2$	$x_1^2 x_2$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$

In Example 2 with $n = 3$ we have

g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^9	$x_1 x_4^2$	$x_1 x_2^4$	$x_1 x_4^2$	$x_1^3 x_2^3$	$x_1^3 x_2^3$	$x_1^3 x_2^3$	$x_1^3 x_2^3$

and so

$$C_G(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1 x_2^4 + 4x_1^3 x_2^3 + 2x_1 x_4^2).$$

Theorem (Polya)

$$PI = C_G\left(\sum_{c\in C} w_c, \sum_{c\in C} w_c^2, \ldots, \sum_{c\in C} w_c^{\Delta}\right).$$

Proof In Example 2, we replace x_1 by $R+B$, x_2 by R^2+B^2 and so on. When $n=2$ this gives

$$\begin{aligned} PI &= \frac{1}{8}((R+B)^4 + 3(R^2+B^2)^2 + \\ &\qquad 2(R+B)^2(R^2+B^2) + 2(R^4+B^4)) \\ &= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \end{aligned}$$

Putting $R=B=1$ gives the number of distinct colorings. Note also the formula for PI tells us that there are 2 distinct colorings using 2 reds and 2 Blues.

Proof of Polya's Theorem

Let $X = X_1 \cup X_2 \cup \cdots \cup X_m$ be the equivalence clases of X under the relation

$$x \sim y \text{ iff } W(x) = W(y).$$

By Lemma 2, $g * x \sim x$ for all $x \in X, g \in G$ and so we can think of G acting on each X_i individually i.e. we use the fact that $x \in X_i$ implies $g * x \in X_i$ for all $i \in [m], g \in G$. We use the notation $g^{(i)} \in G^{(i)}$ when we restrict attention to X_i.

Let m_i denote the number of orbits $\nu_{X_i,G^{(i)}}$ and W_i denote the common PI of $G^{(i)}$ acting on X_i. Then

$$
\begin{aligned}
PI &= \sum_{i=1}^{m} m_i W_i \\
&= \sum_{i=1}^{m} W_i \left(\frac{1}{|G|} \sum_{g \in G} |Fix(g^{(i)})| \right) && \text{by Theorem 2} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i \\
&= \frac{1}{|G|} \sum_{g \in G} W(Fix(g)) && (16)
\end{aligned}
$$

Note that (16) follows from $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ iff $x \in X_i$ and $g * x = x$.

Suppose now that $ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left(\sum_{c \in C} w_c \right)^{k_1} \left(\sum_{c \in C} w_c^2 \right)^{k_2} \cdots \left(\sum_{c \in C} w_c^\Delta \right)^{k_\Delta} . \quad (17)$$

Substituting (17) into (16) yields the theorem.

To verify (17) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of g must be given the same color. A cycle of length i will then contribute a factor $\sum_{c \in C} w_c^i$ where the term w_c^i comes from the choice of color c for every element of the cycle. □

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Game 1 Start with n chips. Players A,B alternately take 1,2,3 or 4 chips until there are none left. The winner is the person who takes the last chip:

Example

	A	B	A	B	A	
$n = 10$	3	2	4	1		B wins
$n = 11$	1	2	3	4	1	A wins

What is the optimal strategy for playing this game?

Game 2 Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') where $0 \le m' < m$ and $0 \le n' < n$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 2a Chip placed at point (m, n). Players can move chip to (m', n) or (m, n') or to $(m - a, n - a)$ where $0 \le m' < m$ and $0 \le n' < n$ and $0 \le a \le \min\{m, n\}$. The player who makes the last move and puts the chip onto $(0, 0)$ wins.

What is the optimal strategy for this game?

Game 3 W is a set of words. A and B alternately remove words $w_1, w_2, \ldots,$ from W. The rule is that the first letter of w_{i+1} must be the same as the last letter of w_i. The player who makes the last legal move wins.

Example

$W = \{England, France, Germany, Russia, Bulgaria, \ldots\}$

What is the optimal strategy for this game?

Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$.
(x, y) is an arc of D iff one can move from position x to position y.

We assume that the digraph is finite and that it is **acyclic** i.e. there are no directed cycles.

The game starts with a token on vertex x_0 say, and players alternately move the token to $x_1, x_2, \ldots,$ where $x_{i+1} \in N^+(x_i)$, the set of out-neighbours of x_i. The game ends when the token is on a **sink** i.e. a vertex of out-degree zero. The last player to move is the winner.

Abstraction

Example 1: $V(D) = \{0, 1, \ldots, n\}$ and $(x, y) \in A$ iff $x - y \in \{1, 2, 3, 4\}$.

Example 2: $V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and $(x, y) \in N^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$.

Example 2a: $V(D) = \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ and $(x, y) \in N^+((x', y')))$ iff $x = x'$ and $y > y'$ or $x > x'$ and $y = y'$ or $x - x' = y - y' > 0$.

Example 3: $V(D) = \{(W', w) : \ W' \subseteq W \setminus \{w\}\}$. w is the last word used and W' is the remaining set of unused words. $(X', w') \in N^+((X, w))$ iff $w' \in X$ and w' begins with the last letter of w. Also, there is an arc from $(W, \cdot)$ to $(W \setminus \{w\}, w)$ for all w, corresponding to the games start.

Abstraction

We will first argue that such a game must eventually end.

A **topological numbering** of digraph $D = (X, A)$ is a map $f : X \to [n]$, $n = |X|$ which satisfies $(x, y) \in A$ implies $f(x) < f(y)$.

Theorem

A finite digraph $D = (X, A)$ is acyclic iff it admits at least one topological numbering.

Proof Suppose first that D has a topological numbering. We show that it is acyclic.

Suppose that $C = (x_1, x_2, \ldots, x_k, x_1)$ is a directed cycle. Then $f(x_1) < f(x_2) < \cdots < f(x_k) < f(x_1)$, contradiction.

Abstraction

Suppose now that D is acyclic. We first argue that D has at least one sink.

Thus let $P = (x_1, x_2, \ldots, x_k)$ be a longest simple path in D. We claim that x_k is a sink.

If D contains an arc (x_k, y) then either $y = x_i, 1 \leq i \leq k-1$ and this means that D contains the cycle $(x_i, x_{i+1}, \ldots, x_k, x_i)$, contradiction or $y \notin \{x_1, x_2, \ldots, x_k\}$ and then (P, y) is a longer simple path than P, contradiction.

Abstraction

We can now prove by induction on n that there is at least one topological numbering.

If $n = 1$ and $X = \{x\}$ then $f(x) = 1$ defines a topological numbering.

Now asssume that $n > 1$. Let z be a sink of D and define $f(z) = n$. The digraph $D' = D - z$ is acyclic and by the induction hypothesis it admits a topological numbering, $f : X \setminus \{z\} \to [n-1]$.

The function we have defined on X is a topological numbering. If $(x, y) \in A$ then either $x, y \neq z$ and then $f(x) < f(y)$ by our assumption on f, or $y = z$ and then $f(x) < n = f(z)$ ($x \neq z$ because z is a sink). $\square$

Abstraction

The fact that D has a topological numbering implies that the game must end. Each move increases the f value of the current position by at least one and so after at most n moves a sink must be reached.

The positions of a game are partitioned into 2 sets:

- P-positions: The next player cannot win. The previous player can win regardless of the current player's strategy.
- N-positions: The next player has a strategy for winning the game.

Thus an N-position is a winning position for the next player and a P-position is a losing position for the next player.

The main problem is to determine N and P and what the strategy is for winning from an N-position.

Abstraction

Let the vertices of D be $x_1, x_2, \ldots, x_n$, in topological order.

Labelling procedure

1. $i \leftarrow n$, Label x_n with P. $N \leftarrow \emptyset$, $P \leftarrow \emptyset$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. Label x_i with N, if $N^+(x_i) \cap P \neq \emptyset$.
4. Label x_i with P, if $N^+(x_i) \subseteq N$.
5. goto 2.

The partition N, P satisfies

$$x \in N \textit{ iff } N^+(x) \cap P \neq \emptyset$$

To play from $x \in N$, move to $y \in N^+(x) \cap P$.

In Game 1, $P = \{5k : \ k \geq 0\}$.

In Game 2, $P = \{(x, x) : \ x \geq 0\}$.

Lemma

The partition into N, P satisfying $x \in N$ iff $N^+(x) \cap P \neq \emptyset$ is unique.

Proof If there were two partitions $N_i, P_i, \ i = 1, 2$, let x_i be the vertex of highest topological number which is not in $(N_1 \cap N_2) \cup (P_1 \cap P_2)$. Suppose that $x_i \in N_1 \setminus N_2$.

But then $x_i \in N_1$ implies $N^+(x_i) \cap P_1 \cap \{x_{i+1}, \ldots, x_n\} \neq \emptyset$ and $x_i \in P_2$ implies $N^+(x_i) \cap P_2 \cap \{x_{i+1}, \ldots, x_n\} = \emptyset$.

But $P_1 \cap \{x_{i+1}, \ldots, x_n\} = P_2 \cap \{x_{i+1}, \ldots, x_n\}$. $\square$

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Sums of games

Suppose that we have p games $G_1, G_2, \ldots, G_p$ with digraphs $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, p$.
The sum $G_1 \oplus G_2 \oplus \cdots \oplus G_p$ of these games is played as follows. A position is a vector $(x_1, x_2, \ldots, x_p) \in X = X_1 \times X_2 \times \cdots \times X_p$. To make a move, a player chooses i such that x_i is not a sink of D_i and then replaces x_i by $y \in N_i^+(x_i)$. The game ends when each x_i is a sink of D_i for $i = 1, 2, \ldots, n$.

Knowing the partitions N_i, P_i for game $i = 1, 2, \ldots, p$ does not seem to be enough to determine how to play the sum of the games.

We need more information. This will be provided by the Sprague-Grundy Numbering

Example

Nim In a one pile game, we start with $a \geq 0$ chips and while there is a positive number x of chips, a move consists of deleting $y \leq x$ chips. In this game the N-positions are the positive integers and the unique P-position is 0.

In general, Nim consists of the sum of n single pile games starting with $a_1, a_2, \ldots, a_n > 0$. A move consists of deleting some chips from a non-empty pile.

Example 2 is Nim with 2 piles.

Sprague-Grundy (SG) Numbering

For $S \subseteq \{0, 1, 2, \ldots, \}$ let

$$mex(S) = \min\{x \geq 0 : \; x \notin S\}.$$

Now given an acyclic digraph $D = X, A$ with topological ordering $x_1, x_2, \ldots, x_n$ define g iteratively by

1. $i \leftarrow n$, $g(x_n) = 0$.
2. $i \leftarrow i - 1$. If $i = 0$ STOP.
3. $g(x_i) = mex(\{g(x) : \; x \in N^+(x_i)\})$.
4. goto 2.

Lemma

$$x \in P \leftrightarrow g(x) = 0.$$

Proof Because

$$x \in N \textit{ iff } N^+(x) \cap P \neq \emptyset$$

all we have to show is that

$$g(x) > 0 \textit{ iff } \exists y \in N^+(y) \textit{ such that } g(y) = 0.$$

But this is immediate from $g(x) = mex(\{g(y) : \; y \in N^+(x)\})$ □

Another one pile subtraction game.

- A player can remove any even number of chips, but not the whole pile.
- A player can remove the whole pile if it is odd.

The terminal positions are 0 or 2.

Lemma

$g(0) = 0$, $g(2k) = k - 1$ *and* $g(2k - 1) = k$ *for* $k \geq 1$.

Proof 0,2 are terminal postions and so $g(0) = g(2) = 0$. $g(1) = 1$ because the only position one can move to from 1 is 0. We prove the remainder by induction on k.

Assume that $k > 1$.

$$\begin{aligned} g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(2)\} \\ &= mex\{k-2, k-3, \ldots, 0\} \\ &= k-1. \\ g(2k-1) &= mex\{g(2k-3), g(2k-5), \ldots, g(1), g(0)\} \\ &= mex\{k-1, k-2, \ldots, 0\} \\ &= k. \end{aligned}$$

□

Sums of games

We now show how to compute the *SG* numbering for a sum of games.

For binary integers $a = a_m a_{m-1} \cdots a_1 a_0$ and $b = b_m b_{m-1} \cdots b_1 b_0$ we define $a \oplus b = c_m c_{m-1} \cdots c_1 c_0$ by

$$c_i = \begin{cases} 1 & \text{if } a_i \neq b_i \\ 0 & \text{if } a_i = b_i \end{cases}$$

for $i = 1, 2, \ldots, m$.

So $11 \oplus 5 = 14$.

Theorem

If g_i is the SG function for game G_i, $i = 1, 2, \ldots, p$ then the SG function g for the sum of the games $G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$ is defined by

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

where $x = (x_1, x_2, \ldots, x_p)$.

For example if in a game of Nim, the pile sizes are $x_1, x_2, \ldots, x_p$ then the *SG* value of the position is

$$x_1 \oplus x_2 \oplus \cdots \oplus x_p$$

Proof It is enough to show this for $p = 2$ and then use induction on p.

Write $G = H \oplus G_p$ where $H = G_1 \oplus G_2 \oplus \cdots \oplus G_{p-1}$. Let h be the SG numbering for H. Then, if $y = (x_1, x_2, \ldots, x_{p-1})$,

$$\begin{aligned} g(x) &= h(y) \oplus g_p(x_p) \qquad \textit{assuming theorem for } p = 2 \\ &= g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_{p-1}(x_{p-1}) \oplus g_p(x_p) \end{aligned}$$

by induction.

It is enough now to show, for $p = 2$, that

- A1 If $x \in X$ and $g(x) = b > a$ then there exists $x' \in N^+(x)$ such that $g(x') = a$.
- A2 If $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq g(x)$.
- A3 If $x \in X$ and $g(x) = 0$ and $x' \in N^+(x)$ then $g(x') \neq 0$

Sums of games

A1. Write $d = a \oplus b$. Then

$$a = d \oplus b = d \oplus g_1(x_1) \oplus g_2(x_2). \tag{18}$$

Now suppose that we can show that either

$$(i)\ d \oplus g_1(x_1) < g_1(x_1) \text{ or } (ii)\ d \oplus g_2(x_2) < g_2(x_2) \text{ or both.} \tag{19}$$

Assume that (i) holds.

Then since $g_1(x_1) = mex(N_1^+(x_1))$ there must exist $x_1' \in N_1^+(x_1)$ such that $g_1(x_1') = d \oplus g_1(x_1)$.

Then from (18) we have

$$a = g_1(x_1') \oplus g_2(x_2) = g(x_1', x_2).$$

Furthermore, $(x_1', x_2) \in N^+(x)$ and so we will have verified A1.

Let us verify (19).

Suppose that $2^{k-1} \leq d < 2^k$.

Then d has a 1 in position k and no higher.

Since $d_k = a_k \oplus b_k$ and $a < b$ we must have $a_k = 0$ and $b_k = 1$.

So either (i) $g_1(x_1)$ has a 1 in position k or (ii) $g_2(x_2)$ has a 1 in position k. Assume (i).

But then $d \oplus g_1(x_1) < g_1(x_1)$ since d “destroys” the kth bit of $g_1(x_1)$ and does not change any higher bit.

Sums of games

A2. Suppose without loss of generality that $g(x_1', x_2) = g(x_1, x_2)$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1') \oplus g_2(x_2) = g_1(x_1) \oplus g_2(x_2)$ implies that $g_1(x_1') = g_1(x_1)$, contradiction. □

A3. Suppose that $g_1(x_1) \oplus g_2(x_2) = 0$ and $g_1(x_1') \oplus g_2(x_2) = 0$ where $x_1' \in N^+(x_1)$.

Then $g_1(x_1) = g_1(x_1')$, contradicting $g_1(x_1) = mex\{g_1(x) : x \in N^+(x_1)\}$.

Sums of games

If we apply this theorem to the game of Nim then if the position x consists of piles of x_i chips for $i = 1, 2, \ldots, p$ then $g(x) = x_1 \oplus x_2 \oplus \cdots \oplus x_p$.

In our first example, $g(x) = x \mod 5$ and so for the sum of p such games we have

$$g(x_1, x_2, \ldots, x_p) = (x_1 \mod 5) \oplus (x_2 \mod 5) \oplus \cdots \oplus (x_p \mod 5).$$

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Geography

Start with a chip sitting on a vertex v of a graph or digraph G. A move consists of moving the chip to a neighbouring vertex.

In edge geography, moving the chip from x to y deletes the edge (x, y). In vertex geography, moving the chip from x to y deletes the vertex x.

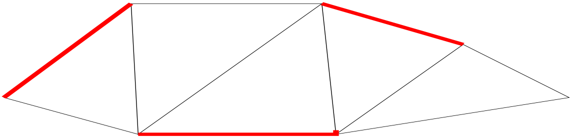
The problem is given a position (G, v), to determine whether this is a P or N position.

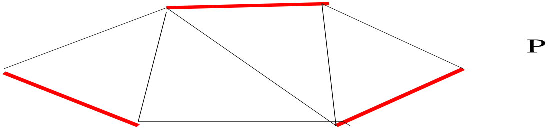
Complexity Both edge and vertex geography are Pspace-hard on digraphs. Edge geography is Pspace-hard on an undirected graph. Only vertex geography on a graph is polynomial time solvable.

Undirected Vertex Geography

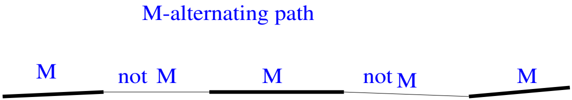
We need some simple results from the theory of matchings on graphs.

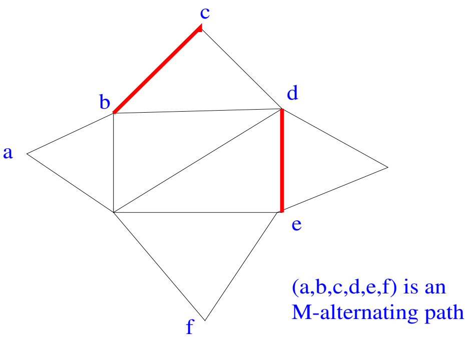
A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





Undirected Vertex Geography





An *M*-alternating path joining 2 *M*-unsaturated vertices is called an *M*-augmenting path.

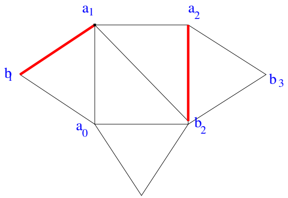
Undirected Vertex Geography

M is a *maximum* matching of G if no matching M' has more edges.

Theorem

M is a maximum matching iff M admits no M-augmenting paths.

Proof Suppose M has an augmenting path
$P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where
$e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and
$f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



Let $M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}$.

Undirected Vertex Geography

- $|M'| = |M| + 1$.
- M' is a matching

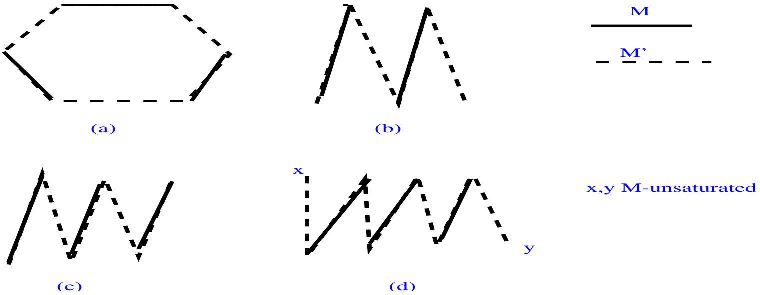
For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

Undirected Vertex Geography

Suppose M is not a maximum matching and $|M'| > |M|$.
Consider $H = G[M \nabla M']$ where $M \nabla M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.
Maximum degree of H is 2 – ≤ 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



$|M'| > |M|$ implies that there is at least one path of type (d).
Such a path is M-augmenting

Undirected Vertex Geography

Theorem

(G, v) is an N-position in UVG iff every maximum matching of G covers v.

Proof (i) Suppose that M is a maximum matching of G which covers v. Player 1's strategy is now: Move along the M-edge that contains the current vertex.

If Player 1 were to lose, then there would exist a sequence of edges $e_1, f_1, \ldots, e_k, f_k$ such that $v \in e_1$, $e_1, e_2, \ldots, e_k \in M$, $f_1, f_2, \ldots, f_k \notin M$ and $f_k = (x, y)$ where y is the current vertex for Player 1 and y is not covered by M.

But then if $A = \{e_1, e_2, \ldots, e_k\}$ and $B = \{f_1, f_2, \ldots, f_k\}$ then $(M \setminus A) \cup B$ is a maximum matching (same size as M) which does not cover v, contradiction.

Undirected Vertex Geography

(ii) Suppose now that there is some maximum matching M which does not cover v. If (v, w) is Player 1's move,then w

must be covered by M, else M is not a maximum matching.

Player 2's strategy is now: Move along the M-edge that contains the current vertex. If Player 2 were to lose then there exists $e_1 = (v, w), f_1, \ldots, e_k, f_k, e_{k+1} = (x, y)$ where y is the current vertex for Player 2 and y is not covered by M.

But then we have defined an augmenting path from v to y and so M is not a maximum matching, contradiction. $\square$

Undirected Vertex Geography

Note that we can determine whether or not v is covered by all maximum matchings as follows: Find the size σ of the maximum matching G.

This can be done in $O(n^3)$ time on an n-vertex graph. Find the size σ' of a maximum matching in $G - v$. Then v is covered by all maximum matchings of G iff $\sigma \neq \sigma'$.

Tic Tac Toe

We consider the following multi-dimensional version of Tic Tac Toe (Noughts and Crosses to the English).

The *board* consists of $[n]^d$. A point on the board is therefore a vector $(x_1, x_2, \ldots, x_d)$ where $1 \leq x_i \leq n$ for $1 \leq i \leq d$.

A *line* is a set points $(x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(d)})$, $j = 1, 2, \ldots, n$ where each sequence $x^{(i)}$ is either (i) of the form $k, k, \ldots, k$ for some $k \in [n]$ or is (ii) $1, 2, \ldots, n$ or is (iii) $n, n-1, \ldots, 1$. Finally, we cannot have Case (i) for all i.

Thus in the (familiar) 3×3 case, the top row is defined by $x^{(1)} = 1, 1, 1$ and $x^{(2)} = 1, 2, 3$ and the diagonal from the bottom left to the top right is defined by $x^{(1)} = 3, 2, 1$ and $x^{(2)} = 1, 2, 3$

Lemma

The number of winning lines in the (n, d) game is $\frac{(n+2)^d - n^d}{2}$.

Proof In the definition of a line there are n choices for k in (i) and then (ii), (iii) make it up to $n + 2$. There are d independent choices for each i making $(n + 2)^d$.

Now delete n^d choices where only Case (i) is used. Then divide by 2 because replacing (ii) by (iii) and vice-versa whenever Case (i) does not hold produces the same set of points (traversing the line in the other direction). □

Tic Tac Toe

The game is played by 2 players. The Red player (X player) goes first and colours a point red. Then the Blue player (0 player) colours a different point blue and so on.

A player wins if there is a line, all of whose points are that players colour. If neither player wins then the game is a draw. The second player does not have a wnning strategy:

Lemma

Player 1 can always get at least a draw.

Proof We prove this by considering *strategy stealing*.

Suppose that Player 2 did have a winning strategy. Then Player 1 can make an arbitrary first move x_1. Player 2 will then move with y_1. Player 1 will now win playing the winning strategy for Player 2 against a first move of y_1.

This can be carried out until the strategy calls for move x_1 (if at all). But then Player 1 can make an arbitrary move and continue, since x_1 has already been made. □

The Hales-Jewett Theorem of Ramsey Theory implies that there is a winner in the (n, d) game, when n is large enough with respect to d. The winner is of course Player 1.

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Tic Tac Toe

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

The above array gives a strategy for Player 2 in the 5×5 game ($d = 2, n = 5$).

For each of the 12 lines there is an associated pair of positions. If Player 1 chooses a position with a number i, then Player 2 responds by choosing the other cell with the number i.

This ensures that Player 1 cannot take line i. If Player 1 chooses the * then Player 2 can choose any cell with an unused number.

Tic Tac Toe

So, later in the game if Player 1 chooses a cell with j and Player 2 already has the other j, then Player 2 can choose an arbitrary cell.

Player 2's strategy is to ensure that after all cells have been chosen, he/she will have chosen one of the numbered cells asociated with each line. This prevents Player 1 from taking a whole line. This is called a *pairing* strategy.

Tic Tac Toe

We now generalise the game to the following: We have a family $\mathcal{F} = A_1, A_2, \ldots, A_N \subseteq A$. A move consists of one player, taking an uncoloured member of A and giving it his colour.

A player wins if one of the sets A_i is completely coloured with his colour.

A pairing strategy is a collection of distinct elements $X = \{x_1, x_2, \ldots, x_{2N-1}, x_{2N}\}$ such that $x_{2i-1}, x_{2i} \in A_i$ for $i \geq 1$.

This is called a *draw forcing pairing*. Player 2 responds to Player 1's choice of $x_{2i+\delta}, \delta = 0, 1$ by choosing $x_{2i+3-\delta}$. If Player 1 does not choose from X, then Player 2 can choose any uncoloured element of X.

Tic Tac Toe

In this way, Player 2 avoids defeat, because at the end of the game Player 2 will have coloured at least one of each of the pairs x_{2i-1}, x_{2i} and so Player 1 cannot have completely coloured A_i for $i = 1, 2, \ldots, N$.

Theorem

If

$$\left|\bigcup_{A\in\mathcal{G}} A\right| \geq 2|\mathcal{G}| \qquad \forall\mathcal{G}\subseteq\mathcal{F} \tag{20}$$

then there is a draw forcing pairing.

Proof We define a bipartite graph Γ. A will be one side of the bipartition and $B=\{b_1,b_2,\ldots,b_{2N}\}$. Here b_{2i-1} and b_{2i} both represent A_i in the sense that if $a\in A_i$ then there is an edge (a,b_{2i-1}) and an edge (a,b_{2i}).

A draw forcing pairing corresponds to a complete matching of B into A and the condition (20) implies that Hall's condition is satisfied. □

Corollary

If $|A_i| \geq n$ for $i = 1, 2, \ldots, n$ and every $x \in A$ is contained in at most $n/2$ sets of $\mathcal{F}$ then there is a draw forcing pairing.

Proof The degree of $a \in A$ is at most $2(n/2)$ in Γ and the degree of each $b \in B$ is at least n. This implies (via Hall's condition) that there is a complete matching of B into A. □

Consider Tic tac Toe when $d = 2$. If n is even then every array element is in at most 3 lines (one row, one column and at most one diagonal) and if n is odd then every array element is in at most 4 lines (one row, one column and at most two diagonals).

Thus there is a draw forcing pairing if $n \geq 6$, n even and if $n \geq 9$, n odd. (The cases $n = 4, 7$ have been settled as draws. $n = 7$ required the use of a computer to examine all possible strategies.)

In general we have

Lemma

If $n \geq 3^d - 1$ and n is odd or if $n \geq 2^d - 1$ and n is even, then there is a draw forcing pairing of (n, d) Tic tac Toe.

Proof We only have to estimate the number of lines through a fixed point $\mathbf{c} = (c_1, c_2, \ldots, c_d)$.

If n is odd then to choose a line L through $\mathbf{c}$ we specify, for each index i whether L is (i) constant on i, (ii) increasing on i or (iii) decreasing on i.

This gives 3^d choices. Subtract 1 to avoid the all constant case and divide by 2 because each line gets counted twice this way.

Tic Tac Toe

When n is even, we observe that once we have chosen in which positions L is constant, L is determined.

Suppose $c_1 = x$ and 1 is not a fixed position. Then every other non-fixed position is x or $n - x + 1$. Assuming w.l.o.g. that $x \leq n/2$ we see that $x < n - x + 1$ and the positions with x increase together at the same time as the positions with $n - x + 1$ decrease together.

Thus the number of lines through $\mathbf{c}$ in this case is bounded by $\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$. □

Quasi-probabilistic method

We now prove a theorem of Erdős and Selfridge.

Theorem

If $|A_i| \geq n$ for $i \in [N]$ and $N < 2^{n-1}$, then Player 2 can get a draw in the game defined by $\mathcal{F}$.

Proof At any point in the game, let C_j denote the set of elements in A which have been coloured with Player j's colour, $j = 1, 2$ and $U = A \setminus C_1 \cup C_2$. Let

$$\Phi = \sum_{i: A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}.$$

Suppose that the players choices are $x_1, y_1, x_2, y_2, \ldots,$. Then we observe that immediately after Player 1's first move, $\Phi < N2^{-(n-1)} < 1$.

Quasi-probabilistic method

We will show that Player 2 can keep $\Phi < 1$ through out. Then at the end, when $U = \emptyset$, $\Phi = \sum_{i:A_i \cap C_2 = \emptyset} 1 < 1$ implies that $A_i \cap C_2 \neq \emptyset$ for all $i \in [N]$.

So, now let Φ_j be the value of Φ after the choice of $x_1, y_1, \ldots, x_j$. then if U, C_1, C_2 are defined at precisely this time,

$$\begin{aligned}
\Phi_{j+1} - \Phi_j &= -\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \notin A_i, x_{j+1} \in A_i}} 2^{-|A_i \cap U|} \\
&\leq -\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y_j \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i:A_i \cap C_2 = \emptyset \\ x_{j+1} \in A_i}} 2^{-|A_i \cap U|}
\end{aligned}$$

Quasi-probabilistic method

We deduce that $\Phi_{j+1} - \Phi_j \leq 0$ if Player 2 chooses y_j to maximise $\sum_{\substack{i:A_i \cap C_2 = \emptyset \\ y \in A_i}} 2^{-|A_i \cap U|}$ over y.

In this way, Player 2 keeps $\Phi < 1$ and obtains a draw. □

In the case of (n, d) Tic Tac Toe, we see that Player 2 can force a draw if

$$\frac{(n+2)^d - n^d}{2} < 2^{n-1}$$

which is implied, for n large, by

$$n \geq (1+\epsilon) d \log_2 d$$

where $\epsilon > 0$ is a small positive constant.

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Hereditary Families

Given a Ground Set E, a Hereditary Family $\mathcal{A}$ on E is collection of subsets $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ (the independent sets) such that

$$I \in \mathcal{I} \text{ and } J \subseteq I \text{ implies that } J \in \mathcal{I}.$$

1. The set $\mathcal{M}$ of matchings of a graph $G = (V, E)$.
2. The set of (edge-sets of) forests of a graph $G = (V, E)$.
3. The set of stable sets of a graph $G = (V, E)$. We say that S is stable if it contains no edges.
4. If $G = (A, B, E)$ is a bipartite graph and $\mathcal{I} = \{S \subseteq B : \exists \text{ a matching } M \text{ that covers } S\}$.
5. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be the columns of an $m \times n$ matrix $\overline{A}$. Then $E = [n]$ and $\mathcal{I} = \{S \subseteq [n] : \{\mathbf{c}_i, i \in S\} \text{ are linearly independent}\}$.

Matroids

An independence system is a matroid if whenever $I, J \in \mathcal{I}$ with $|J| = |I| + 1$ there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$. We call this the Independent Augmentation Axiom – IAA.

Matroid independence is a generalisation of linear independence in vector spaces. Only Examples 2,4 and 5 above are matroids.

To check Example 5, let $\overline{A}_I$ be the $m \times |I|$ sub-matrix of $\overline{A}$ consisting of the columns in I. If there is no $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then $\overline{A}_J = \overline{A}_I \mathbf{M}$ for some $|I| \times |J|$ matrix $\mathbf{M}$.

Matrix $\mathbf{M}$ has more columns than rows and so there exists $\mathbf{x} \neq 0$ such that $\mathbf{Mx} = \mathbf{0}$. But then $\overline{A}_J \mathbf{x} = \mathbf{0}$, implying that the columns of $\overline{A}_J$ are linearly dependent. Contradiction.

These are called Representable Matroids.

Cycle Matroids/Graphic Matroids

To check Example 2 we define the vertex-edge incidence matrix $\overline{A}_G$ of graph $G = (V, E)$ over GF_2.

$\overline{A}_G$ has a row for each vertex $v \in V$ and a column for each edge $e \in E$. There is a 1 in row v, column e iff $v \in e$.

We verify that a set of columns $\mathbf{c}_i, i \in I$ are linearly dependent iff the corresponding edges contain a cycle.

If the edges contain a cycle $(v_1, v_2, \ldots, v_k, v_1)$ then the sum of the columns corresponding to the vertices of the cycle is $\mathbf{0}$.

To show that a forest F defines a linearly independent set of columns I_F, we use induction on the number of edges in the forest. This is trivial if $|E(F)| = 1$.

Cycle Matroids/Graphic Matroids

Let $\overline{A}_F$ denote the submatrix of $\overline{A}$ made up of the columns corresponding F.

Now a forest F must contain a vertex v of degree one. This means that the row corresponding to v in $\overline{A}_F$ has a single one, in column e say.

Consider the forest $F' = F \setminus \{e\}$. Its corresponding columns $I_{F'}$ are linearly independent, by induction. Adding back e adds a row with a single one and preserves independence. Let $\mathbf{B}$ denote $\overline{A}_{F'}$ minus row e.

$$\overline{A}_F = \begin{bmatrix} 1 & \mathbf{0} \\ & \mathbf{B} \end{bmatrix}.$$

We now check Example 4. These are called Transversal Matroids. If M_1, M_2 are two matchings in a graph G then $M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ consists of alternating paths and cycles.

Suppose now that we have two matchings M_1, M_2 in bipartite graph $G = (A, B, E)$. Let $I_j, j = 1, 2$ be the vertices in B covered by M_j. Suppose that $|I_1| > |I_2|$.

Then $M_1 \oplus M_2$ must contain an alternating path P with end points $b \in I_1 \setminus I_2, a \in A$. Let E_1 be the M_1 edges in P and let E_2 be the M_2 edges of P. Then $(M_1 \cup E_1) \setminus E_2$ is a matching that covers $I_1 \cup \{b\}$.

Representable Matroids

A matroid is binary if is representable by a matrix over GF_2.

So a graphic matroid is binary.

A matroid is regular if it can be represented by a matrix of elements in $\{0, \pm 1\}$ for which every square sub-matrix has determinant $0, \pm 1$. These are called totally unimodular matrices

A matrix with 2 non-zeros in each column, one equal to +1 and the other equal to -1 is totally unimodular. This implies that graphic matroids are regular. (Take the vertex-edge incidence matrix and replace one of the ones in each column by a -1.)

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Bases

Theorem

A collection $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ of subsets of E form the bases of a matroid on E iff for all i, j and $e \in B_i \setminus B_j$ there exists $f \in B_j \setminus B_i$ such that $(B_i \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.

Proof: Suppose first that $\mathcal{B}$ are the bases of a matroid with independent sets $\mathcal{I}$ and that $e \in B_i$ and $e \notin B_j$. Then $B_i' = B_i \setminus \{e\} \in \mathcal{I}$ and $|B_i'| < |B_j|$. So there exists $f \in B_j \setminus B_i'$ such that $B_i'' = B_i' \cup \{f\} \in \mathcal{I}$. Now $f \neq e$ since $e \notin B_j$ and $|B_i''| = |B_i|$. So B_i'' must be a basis.

Rank

If $S \subseteq E$ then its rank

$$r(S) = \max |\{I \in \mathcal{I} : I \subseteq S\}| .$$

So $S \in \mathcal{I}$ iff $r(S) = |S|$. We show next that r is submodular.

Theorem

If $S, T \subseteq E$ then $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$.

Proof: Let I_1 be a maximal independent subset of $S \cap T$ and let I_2 be a maximal independent subset of $S \cup T$ that contains I_2. (Such a set exists because of the IAA.)

But then

$$r(S \cap T) + r(S \cup T) = |I_1| + |I_2| = |I_2 \cap S| + |I_2 \cap T| \leq r(S) + r(T).$$

Rank

For representable matroids this coresponds to the usual definition of rank.

For the cycle matroid of graph $G = (V, E)$, if $S \subseteq E$ is a set of edges and G_S is the graph (V, S) then $r(S) = |V| - \kappa(G_S)$, where $\kappa(G_S)$ is the number of components of G_S.

This clearly true for connected graphs and so if $C_1, C_2, \ldots, C_s$ are the components of G_S then $r(S) = \sum_{i=1}^{s} |C_i| - 1 = |V| - s$.

For a partition matroid as defined above,

$$r(S) = \sum_{i=1}^{m} \min\{k_i, |S \cap E_i|\}.$$

Circuits

A circuit of a matroid $\mathcal{M}$ is a minimal dependent set. If a set $S \subseteq E, S \notin \mathcal{I}$ then S contains a circuit.

So the circuits of the cycle matroid of a graph G are the cycles.

Theorem

If C_1, C_2 are circuits of $\mathcal{M}$ and $e \in C_1 \cap C_2$ then there is a circuit $C \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Proof: We have $r(C_i) = |C_i| - 1, i = 1, 2$. Also, $r(C_1 \cap C_2) = |C_1 \cap C_2|$ since $C_1 \cap C_2$ is a proper subgraph of C_1.

If $C' = (C_1 \cup C_2) \setminus \{e\}$ contains no circuit then $r(C_1 \cup C_2) \geq r(C') = |C_1 \cup C_2| - 1$. But then

$$|C_1 \cup C_2| - 1 \leq r(C_1 \cup C_2) \leq r(C_1) + r(C_2) - r(C_1 \cap C_2)$$
$$= (|C_1| - 1) + (|C_2| - 1) - |C_1 \cap C_2|.$$

Contradiction. □

Theorem

If B is a basis of $\mathcal{M}$ and $e \in E \setminus B$ then $B' = B \cup \{e\}$ contains a unique circuit $C(e, B)$. Furthermore, if $f \in C(e, B)$ then $(B \cup \{e\}) \setminus \{f\}$ is also a basis of $\mathcal{M}$.

Proof: $B' \notin \mathcal{I}$ because B is maximal. So B' must contain at least one circuit.

Suppose it contains distinct circuits C_1, C_2. Then $e \in C_1 \cap C_2$ and so B' contains a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

But then $C_3 \subseteq B$, contradiction. □

11/19/2021

Dual Matroid

Theorem

If $\mathcal{B}$ denotes the set of bases of a matroid $\mathcal{M}$ on ground set E then $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ is the set of bases of a matroid $\mathcal{M}^*$, the dual matroid.

Proof: Suppose that $B_1^*, B_2^* \in \mathcal{B}^*$ and $e \in B_1^* \setminus B_2^*$.

Let $B_i = E \setminus B_i^*, i = 1, 2$. Then $e \in B_2 \setminus B_1$.

So there exists $f \in B_1 m^\star B_2$ such that $(B_2 \cup \{e\}) m^\star \{f\} \in \mathcal{B}$.

This implies that $(\mathcal{B}_2^* \cup \{f\}) m^\star \{e\} \in \mathcal{B}^*$. $\square$

Suppose that each $e \in E$ is given a weight w_e and that the weight $w(I)$ of an independent set I is given by $w(I) = \sum_{e \in I} c_e$. The problem we discuss is

$$\text{Maximize } w(I) \text{ subject to } I \in \mathcal{I}.$$

Greedy Algorithm:

begin

Sort $E = \{e_1, e_2, \ldots, e_m\}$ so $w(e_i) \geq w(e_{i+1})$ for $1 \leq i < m$;

$S \leftarrow \emptyset$;

for $i = 1, 2, \ldots, m$;

begin

if $S \cup \{e_i\} \in \mathcal{I}$ **then**;

begin;

$S \leftarrow S \cup \{e_i\}$;

end;

end;

end

Greedy Algorithm

Theorem

The greedy algorithm finds a maximum weight independent set **for all choices of** w *if and only if it is a matroid.*

Suppose first that the Greedy Algorithm always finds a maximum weight independent set. Suppose that $\emptyset \neq I, J \in \mathcal{I}$ with $|J| = |I| + 1$. Define

$$w(e) = \begin{cases} 1 + \frac{1}{2|I|} & e \in I. \\ 1 & e \in J \setminus I. \\ 0 & e \notin I \cup J. \end{cases}$$

If there does not exist $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then the Greedy Algorithm will choose the elements of I and stop. But I does not have maximum weight. Its weight is $|I| + 1/2 < |J|$. So if Greedy succeeds, then the IAA holds.

Greedy Algorithm

Conversely, suppose that our independence system is a matroid. We can assume that $w(e) > 0$ for all $e \in E$. Otherwise we can restrict ourselves to the matroid defined by $\mathcal{I}' = \{I \subseteq E^+\}$ where $E^+ = \{e \in E : w(e) > 0\}$.

Suppose now that Greedy chooses $I_G = e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ where $i_t < i_{t+1}$ for $1 \leq t < k$. Let $I = e_{j_1}, e_{j_2}, \ldots, e_{j_\ell}$ be any other independent set and assume that $j_t < j_{t+1}$ for $1 \leq t < \ell$. We can assume that $\ell \geq k$, for otherwise we can add something from I_G to I to give it larger weight.

We show next that $k = \ell$ and that $i_t \leq j_t$ for $1 \leq t \leq k$. This implies that $w(I_G) \geq w(I)$.

Greedy Algorithm

Suppose then that there exists t such that $i_t > j_t$ and let t be as small as possible for this to be true.

Now consider $I = \{e_{i_s} : s = 1, 2, \ldots, t-1\}$ and $J = \{e_{j_s} : s = 1, 2, \ldots, t\}$. Now there exists $e_{j_s} \in J \setminus I$ such that $I \cup \{e_{j_s}\} \in \mathcal{I}$.

But $j_s \leq j_t < i_t$ and Greedy should have chosen e_{j_s} before choosing $e_{i_{t+1}}$.

Also, $i_k \leq j_k$ implies that $k = \ell$. Otherwise Greedy can find another element from $I \setminus I_G$ to add.

Minors

Given a graph $G = (V, E)$ and an edge e we can get new graphs by deleting e or contracting e.

We describe a corresponding notion for matroids. Suppose that $F \subseteq E$ then we define the matroid $\mathcal{M}_{\setminus F}$ with independent sets $\mathcal{I}_{\setminus F}$ obtained by deleting F: $I \in \mathcal{I}_{\setminus F}$ if $I \in \mathcal{I},\ I \cap F = \emptyset$.

It is clear that the IAA holds for $\mathcal{M}_{\setminus F}$ and so it is a matroid.

For contraction we will assume that $F \in \mathcal{I}$. Then contracting F defines $\mathcal{M}.F$with independent sets
$\mathcal{I}.F = \{I \in \mathcal{I} : I \cap F = \emptyset,\ I \cup F \in \mathcal{I}\}$.

We argue next that $\mathcal{M}.F$ is also a matroid.

Lemma

$\mathcal{M}.F = (\mathcal{M}^*_{\setminus F})^*$ *and* $\mathcal{M}_{\setminus F} = (\mathcal{M}^*.F)^*$.

Proof:

$$\begin{aligned} I \in \mathcal{I}.F &\leftrightarrow \exists B \in \mathcal{B}_{\setminus F}, I \subseteq B \\ &\leftrightarrow \exists B^* \in \mathcal{B}^*_{\setminus F}, I \cap B^* = \emptyset \\ &\leftrightarrow I \in (\mathcal{I}^*_{\setminus F})^*. \end{aligned}$$

For the second claim we use

$$\mathcal{M}^*.F = (\mathcal{M}^{**}_{\setminus F})^* = (\mathcal{M}_{\setminus F})^*.$$

□

11/22/2021

Suppose we are given two matroids $\mathcal{M}_1, \mathcal{M}_2$ on the same ground set E with $\mathcal{I}_1, \mathcal{I}_2$ and r_1, r_2 etc. having there obvious meaning.

An intersection is a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. We give a min-max relation for the size of the largest independent intersection. Let $\mathcal{J}$ denote the set of intersections.

Theorem (Edmonds)

$$\max\{J \in \mathcal{J}\} = \min\{r_1(A) + r_2(E \setminus A) : A \subseteq E\}.$$

Matroid Intersection

Before proving the theorem let us see a couple of applications:

Hall's Theorem: suppose we are given a bipartite graph $G = (A, B, E)$. Let $\mathcal{M}_A, \mathcal{M}_B$ be the following two partition matroids.

For $\mathcal{M}_A$ we define the partition $E_a = \{e \in x : a \in e\}$, $a \in A$. We let $k_a = 1$ for $a \in A$. We define $\mathcal{M}_B$ similarly.

Intersections correspond to matchings and $r_1(A)$ is the number of vertices in A that are incident with an edge of A. Similarly $r_2(E \setminus A)$ is the number of vertices in B that are incident with an edge not in A.

Matroid Intersection

For $X \subseteq A$, let

$$A_X = \{v \in A : v \in e \text{ for some } e \in X\}.$$

Define B_X similarly.

So

$$\max\{|M|\} = \min\{|A_X| + |B_{E \setminus X}| : \ X \subseteq E\}.$$

Now we can assume that if $e \in E \setminus X$ then $e \cap A_X = \emptyset$, otherwise moving e to X does not increase the RHS of the above.

Let $S = A \setminus A_X$. Then $|B_{E \setminus X}| = |N(A)|$ and so

$$\max\{|M|\} = \min\{|A| - |S| + |N(S)| : S \subseteq A\}.$$

Matroid Intersection

Rainbow Spanning Trees: we are given a connected graph $G = (V, E)$ where each edge $e \in E$ is given a color $c(e) \in [m]$ where $m \geq n - 1$. Let $E_i = \{e : c(e) = i\}$ for $i \in [m]$.

A set of edges S is said to be rainbow colored if $e, f \in S$ implies that $c(e) \neq c(f)$.

For a set $A \subseteq E$, we let

$$r_1(A) = c(A) = |\{i \in [m] : \exists e \in A \ s.t. \ c(e) = i\}|$$
$$r_2(E \setminus A) = n - \kappa(G \setminus A).$$

So, G contains a rainbow spanning tree iff

$$c(A) + (n - \kappa(G \setminus A)) \geq n - 1 \text{ for all } A \subseteq E. \tag{21}$$

Matroid Intersection

We simplify (21) to obtain

$$c(A) + 1 \geq \kappa(G \setminus A). \tag{22}$$

We can then further simplify (22) as follows: if we add to A all edges that use a color used by some edge of A then we do not change $c(A)$ but we do not decrease $\kappa(G \setminus A)$.

Thus we can restrict our sets A to $E_I = \bigcup_{i \in I} E_i$ for some $I \subseteq [m]$. Then (22) becomes

$$\kappa(E_{[m] \setminus I}) \leq |I| + 1 \text{ for all } I \subseteq [m]$$

or

$$\kappa(E_I) \leq m - |I| + 1 \text{ for all } I \subseteq [m]$$

If you think for a moment, you will see that this is obviously necessary.

Matroid Intersection

Proof of the matroid intersection theorem.

For the upper bound consider $J \in \mathcal{J}$ and $A \subseteq E$. Then

$$|J| = |J \cap A| + |J \setminus A| \leq r_1(A) + r_2(E \setminus A).$$

We assume that $e \in \mathcal{J}$ for all $e \in E$. (Loops can be “ignored”.)

We proceed by induction on $|E|$. Let

$$k = \min\{r_1(A) + r_2(E \setminus A) : A \subseteq E\}.$$

Suppose that $|J| < k$ for all $J \in \mathcal{J}$.

Matroid Intersection

Then $(\mathcal{M}_1)_{\setminus\{e\}}$ and $(\mathcal{M}_2)_{\setminus\{e\}}$ have no common independent set of size k. This implies that if $F = E \setminus \{e\}$ then

$$r_1(A) + r_2(F \setminus A) \leq k - 1 \text{ for some } A \subseteq F.$$

Similarly, $\mathcal{M}_1.\{e\}$ and $\mathcal{M}_2.\{e\}$ have no common independent set of size $k - 1$. This implies that

$$r_1(B) - 1 + r_2(E \setminus (B \setminus \{e\})) - 1 \leq k - 2 \text{ for some } e \in B \subseteq E.$$

This gives

$$r_1(A) + r_2(E \setminus (A \cup \{e\})) + r_1(B) + r_2(E \setminus (B \setminus \{e\})) \leq 2k - 1.$$

Matroid Intersection

So, using submodularity and

$$(E \setminus (A \cup \{e\})) \cup (E \setminus (B \setminus \{e\})) = E \setminus (A \cap B)$$

and

$$(E \setminus (A \cup \{e\})) \cap (E \setminus (B \setminus \{e\})) = E \setminus (A \cup B).$$

We have used $e \notin A$ and $e \in B$ here. So,

$$r_1(A \cup B) + r_2(E \setminus (A \cup B)) + r_1(A \cap B) + r_2(E \setminus (A \cap B)) \\ \leq 2k - 1.$$

But, by assumption,

$$r_1(A \cup B) + r_2(E \setminus (A \cup B)) \geq k, \;\; r_1(A \cap B) + r_2(E \setminus (A \cap B)) \geq k,$$

contradiction.