Class 12

Systems of distinct representatives Let $A_1, A_2, \ldots, A_m$ be subsets of A. An SDR (system of distinct representatives) is a collection of distinct elements $a_1, a_2, \ldots, a_m$ such that $a_i \in A_i$ for $i = 1, 2, \ldots, m$.

Hall's Theorem implies the following: $A_1, A_2, \ldots, A_m$ has an SDR iff

$$\left|\bigcup_{i \in I} A_i\right| \geq |I|, \qquad \forall I \subseteq [m].$$

Just define a bipartit egraph with **vertices** $X = \{A_1, A_2, \ldots, A_m\}$ and $Y = A$ and an edge (A_i, a) iff $a \in A_i$.

König's Theorem Let $G = (V, E)$ be a grap. A **vertex cover** is a set $S \subseteq V$ such that if $e = (x, y)$ is an edge of G then either $x \in S$ or $y \in S$ (or both).

If M is a matching of G and S is a vertex cover, then

$$|M| \leq |S|.$$

This is because a single vertex covers only one edge of M.

Theorem 1. *In a bipartite graph,*

$$\max_{M \text{ matching}} |M| = \min_{S \text{ cover}} |S|.$$

Proof Suppose that $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and $S = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ is a minimum cover. We show that there exists a matching of size $r + s$.

Let H be the bipartite subgraph of G induced by $X' = \{x_1, x_2, \ldots, x_r\}$ and $Y' = Y = \{y_{s+1}, y_{s+2}, \ldots, y_n\}$.

Suppose that $A \subseteq X'$ and $|\Gamma_H(A)| < |A|$. The we would get a smaller cover of G by replacing A with $\Gamma_H(A)$. Thus $|\Gamma_H(A)| \geq |A|$ and Hall's theorem implies that there is a matching M_1 in H of X' into Y'. Similarly there is a matching M_2 of $\{y_1, y_2, \ldots, y_s\}$ into $\{x_{r+1}, \ldots, x_n\}$. The matching $M_1 \cup M_2$ is of size $r + s$. $\square$