

21-325 (Fall 2008): Homework 5 (TWO side)

Due by October 8, in class

Show FULL JUSTIFICATION for all your answers.

1. A group of N people throw their hats into the center of a room. The hats are mixed up, and each person randomly selects one. Find the expected number of people that select their own hat.

Solution. Let X_i be the indicator that the i th person gets his own hat, i.e., $X_i = 1$ if the i th person gets his own hat and $X_i = 0$ if the i th person does not get his own hat. Then the number of people that select their own hat is $\sum_{i=1}^N X_i$. Its expectation is

$$E\left[\sum_{i=1}^N X_i\right] = \sum_{i=1}^N P(X_i = 1) = \sum_{i=1}^N \frac{1}{N} = 1.$$

2. On a given day, your golf score takes values from the range 101 to 110, with probability 0.1, independently from other days. Determined to improve your score, you decide to play on three different days and declare as your score the minimum X of the scores X_1 , X_2 and X_3 on the different days.

(a) Calculate the PMF of X .

Solution. X has range 101 to 110. For $n = 101, \dots, 110$,

$$\begin{aligned} P(X \geq n) &= P(X_1 \geq n, X_2 \geq n, X_3 \geq n) \\ &= P(X_1 \geq n)P(X_2 \geq n)P(X_3 \geq n) \\ &= \left(\frac{110 - n + 1}{10}\right)^3 \end{aligned}$$

Then $P(X = n) = P(X \geq n) - P(X \geq n + 1) = \left(\frac{110 - n + 1}{10}\right)^3 - \left(\frac{110 - n}{10}\right)^3$.

(b) Calculate the expected value of X .

Solution. $E[X] = \sum_{n=101}^{110} n * \left(\left(\frac{110 - n + 1}{10}\right)^3 - \left(\frac{110 - n}{10}\right)^3 \right)$.

3. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2 , calculate the conditional distribution of X given that $X + Y = n$.

Solution. Since $X + Y = n$ and X, Y are nonnegative, then X can choose $1, \dots, n$. So we just need to compute $P(X = i | X + Y = n)$.

$$P(X = i | X + Y = n) = \frac{P(X = i, X + Y = n)}{P(X + Y = n)}$$

$$\begin{aligned}
&= \frac{P(X = i, Y = n - i)}{P(X + Y = n)} \\
&= \frac{P(X = i)P(Y = n - i)}{P(X + Y = n)} \\
&= \frac{P(X = i)P(Y = n - i)}{\sum_{m=0}^n P(X = m)P(Y = n - m)} \\
&= \frac{e^{-\lambda_1} \frac{\lambda_1^i}{i!} e^{-\lambda_2} \frac{\lambda_2^{n-i}}{(n-i)!}}{\sum_{m=0}^n e^{-\lambda_1} \frac{\lambda_1^m}{m!} e^{-\lambda_2} \frac{\lambda_2^{n-m}}{(n-m)!}} \\
&= \frac{\frac{\lambda_1^i}{i!} e^{-\lambda_2} \frac{\lambda_2^{n-i}}{(n-i)!}}{\sum_{m=0}^n e^{-\lambda_1} \frac{\lambda_1^m}{m!} e^{-\lambda_2} \frac{\lambda_2^{n-m}}{(n-m)!}} \\
&= \binom{n}{i} \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} \right]^i \left[\frac{\lambda_2}{\lambda_1 + \lambda_2} \right]^{n-i}.
\end{aligned}$$

So the conditional distribution of X given that $X + Y = n$ is the binomial distribution with parameter $\frac{\lambda_1}{\lambda_1 + \lambda_2}$.

4. Suppose that the number of events that occur in a given time period is a poisson random variable with parameter λ . If each event is classified as a type i event with probability p_i , $i = 1, 2, 3$, $p_1 + p_2 + p_3 = 1$, independently of other events, show that the numbers of type i events that occur, $i = 1, 2, 3$ are independent Poisson random variables with respective parameters λp_i , $i = 1, 2, 3$.

Solution. Please refer to the example of the people entering the post office in the class notes.

5. A random variable X is called a negative Binomial random variable with parameters (r, p) if

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r} \quad n = r, r+1, \dots.$$

Show that $X_1 + X_2 + \dots + X_r$ has a negative Binomial distribution when the X_i , $i = 1, 2, \dots, r$ are independent and they have the same Geometric distribution with parameter p .

Solution. The negative Binomial random variable with parameters (r, p) describes the distribution of the number of trials needed to get r successes. If we let X_1 be the number of trials needed to get the first success, let X_2 be the number of trials needed to get the second success after the first success, and so on. At last, let X_r be the number of trials needed to get the r th success after the $r - 1$ st success. Each X_i has the Geometric distribution with parameter p , X_i , $i = 1, 2, \dots, r$ are independent, and $X_1 + X_2 + \dots + X_r$ is the number of trials needed to get r successes. So $X_1 + X_2 + \dots + X_r$ is a negative Binomial random variable with parameters (r, p) .

6. Suppose that X and Y are independent Geometric random variables with the same parameter p , what is the value of $P(X = i|X + Y = n)$?

Solution.

$$\begin{aligned} P(X = i|X + Y = n) &= \frac{P(X = i)P(Y = n - i)}{P(X + Y = n)} \\ &= \frac{(1 - p)^{i-1}p(1 - p)^{n-i-1}p}{\sum_{m=1}^n (1 - p)^{m-1}p(1 - p)^{n-m-1}p} \\ &= \frac{1}{n}. \end{aligned}$$

7. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number Y at random from the subset $\{1, \dots, X\}$.

(a) Find the joint PMF of X and Y .

Solution. Let $1 \leq m \leq n \leq 5$. $P(X = n, Y = m) = P(X = n)P(Y = m|X = n) = \frac{1}{5} \frac{1}{n} = \frac{1}{5n}$.

(b) Find the conditional PMF of X given Y . Are X and Y independent? Why?

Solution. Let $1 \leq m \leq n \leq 5$. $P(X = n|Y = m) = \frac{P(X=n, Y=m)}{P(Y=m)} = \frac{P(X=n, Y=m)}{\sum_{n=m}^5 P(X=n, Y=m)} = \frac{1/n}{\sum_{n=m}^5 1/n}$. Hence X and Y are not independent since $P(X = n|Y = m) \neq P(X = n)$.

(c) Find $E[Y|X]$, and use that to calculate $E[Y]$.

Solution. $E[Y|X] = \sum_{n=1}^5 E[Y|X = n]1_{\{X=n\}} = \sum_{n=1}^5 \frac{1+n}{2} 1_{\{X=n\}}$. Then $E[Y] = \sum_{n=1}^5 \frac{1+n}{2} P(X = n) = \sum_{n=1}^5 \frac{1+n}{2} \frac{1}{5} = 2.8$. You pick a number between 1 and 6. Then you roll three fair, independent dice.

(1) If your number never comes up, then you lose a dollar.

(2) If your number comes up once, then you win a dollar.

(3) If your number comes up twice, then you win two dollars.

(4) If your number comes up three times, you win four dollars!

What is your expected payoff? Is playing this game likely to be profitable for you or not?

Solution. Let the random variable R be the amount of money won or lost by the player in a round. We can compute the expected value of R as follows:

$$\begin{aligned} E[R] &= -1 * P(0 \text{ matches}) + 1 * P(1 \text{ match}) + 2 * P(2 \text{ matches}) + 4 * P(3 \text{ matches}) \\ &= -1(5/6)^3 + 1 * 3 * (1/6) * (5/6)^2 + 2 * 3 * (1/6)^2(5/6) + 4 * (1/6)^3 \\ &= \frac{-16}{216} \end{aligned}$$

You can expect to lose $16/216$ of a dollar (about 7.4 cents) in every round. This is a horrible game!