1. Find the mean, variance, probability generating function and characteristic function of a $\mathrm{Bin}(n,p)$ random variable. Find the limit of its probability generating function as well as characteristic function as $n \to \infty$ with $np \to \lambda$. What is the distribution of the sum of independent Poisson random variables?

2. Find the probability generating function of a Poisson random variable with parameter λ. Let $X_1, \ldots, X_n$ be independent random variables with the Poisson distribution, each with parameter 1. Find the probability generating function of $S_n = X_1 + \ldots + X_n$. What is the distribution of S_n? What is the mean and variance of S_n? Prove that for positive t, $\mathbb{P}\left(S_n \geq (1+t)n\right) \leq \frac{1}{t^2 n}$. Show that $\lim_{n\to\infty} e^{-n} \sum_{k \geq 1.1n} \frac{n^k}{k!} = 0$.

3. Fix $p \in (0,1)$. Let S_n be a random variable with the binomial distribution with parameters n and p. Show that for every positive ε, $\lim_{n\to\infty} \mathbb{P}\left(S_n > (p+\varepsilon)n\right) = 0$. Does the sequence $\frac{S_n}{n}$ converge i) a.s., ii) in probability, iii) in L_2 iv) in distribution?

4. Let X be a random variable with density $f(x) = \frac{1}{2}e^{-|x|}$. Find $\mathbb{E}X$ and $\mathbb{E}|X|$. Find its variance. Find the distribution function of $|X|$, εX and $\varepsilon + X$ and sketch their plots (ε is an independent of X random sign). Find the distribution function of X^2.

5. Let X and Y be independent standard Gaussian random variables and let a, b, c, d be real numbers. What is the distribution of $aX + bY$? Find $\mathrm{Cov}(aX + bY, cX + dY)$. Show that $aX + bY$ and $cX + dY$ are independent if and only if the vectors (a, b) and (c, d) are orthogonal. Find the density of $\sqrt{X^2 + Y^2}$.

6. What is the density of a standard Gaussian random variable, that is a Gaussian random variable with mean zero and variance one? Let X and Y be independent standard Gaussian random variables. What is the distribution of $\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$? Are the variables X and $X+Y$ independent? Are the variables $\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$ and $\frac{\sqrt{3}}{2}X + \frac{1}{2}Y$ independent?

7. Let S_n be the number of heads after throwing n times a biased coin showing heads with probability $1/3$. What is the mean and variance of S_n? Show that

$$\lim_{n\to\infty} \mathbb{P}\left(S_n > n/3 + \sqrt{n}\right) = \int_{\frac{3}{\sqrt{2}}}^{\infty} e^{-x^2/2} \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

8. Let S_n be the number of ones when throwing a fair die n times. What is the limit of $\mathbb{P}\left(S_n > n/6 + \sqrt{n}\right)$? Let S be the number of ones when throwing a fair die 18000 times. Find a good approximation to $\mathbb{P}(2950 < S < 3050)$. How can you bound the error you make?

9. Let f be a continuous function on $[0,1]$. Find $\lim_{n\to\infty} \int_0^1 \dots \int_0^1 f\left(\frac{x_1+\dots+x_n}{n}\right) \mathrm{d}x_1 \dots \mathrm{d}x_n$ (or show it does not exist).

10. Let f be a continuous function on $[0,1]$. Find $\lim_{n\to\infty} \int_0^1 \dots \int_0^1 f(\sqrt[n]{x_1 \cdot \ldots \cdot x_n}) \mathrm{d}x_1 \dots \mathrm{d}x_n$ (or show it does not exist).

11. Let $v_1, \dots, v_m$ be unit vectors in $\mathbb{R}^n$. Show that there is a choice of signs $\varepsilon_1, \dots, \varepsilon_m$ such that the vector $\varepsilon_1 v_1 + \dots + \varepsilon_m v_m$ has length at least $\sqrt{m}$.

12. Let g be a standard Gaussian random variable. Find $\mathbb{E} g^{2n}$.

13.* Let $\varepsilon_1, \varepsilon_2, \dots$ be independent random signs. Let $X_n = \frac{2}{n} \sum_{1 \le i < j \le n} \varepsilon_i \varepsilon_j$. Does the sequence X_n converge in distribution? If yes, find its limit.