Probability 21-325 **Midterm 2 (19th April 2019)** **TT**

Please write down your name in CAPITAL letters. No resources allowed (books, notes, electronic devices, etc.)

1. For $\alpha \in \mathbb{R}$, consider the function

$$F_\alpha(t) = \begin{cases} 0, & t < -1, \\ \alpha(t-1) + \frac{1}{2}, & -1 \le t < 1, \\ 1, & t \ge 1. \end{cases}$$

Find all α such that F_α is the distribution function of a random variable. For those α, let X_α be a random variable with the distribution function F_α. Find $\mathbb{P}(X_\alpha = -1)$, $\mathbb{P}(X_\alpha = 1)$ and $\mathbb{P}(X_\alpha > 0)$. Is X_α a continuous random variable? Find the distribution function of $Y = X_0^2$.

2. Let X and Y be independent random variables such that X is uniformly distributed on $[-1, 1]$ and Y has the exponential distribution with parameter 1. Find $\mathbb{E}[(X+Y)^2]$.

3. Let X and Y be independent standard Gaussian random variables. Let $Z = 2X - Y$. Is Z a Gaussian random variable? Find the mean and variance of Z. Find the density of Z. Consider the random vector $V = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$. Is V a Gaussian random vector? Find the density of V. Does V have independent coordinates?

4. Let G be a standard Gaussian vector in $\mathbb{R}^2$. Let u and v be unit vectors in $\mathbb{R}^2$. Show that

$$\mathbb{E}\big[\, \langle u, G\rangle \, \langle v, G\rangle \,\big] = \langle u, v\rangle$$

and

$$\mathbb{E}\big[\mathrm{sgn}(\langle u, G\rangle)\mathrm{sgn}(\langle v, G\rangle)\big] = \frac{2}{\pi}\arcsin(\langle u, v\rangle).$$

Here $\langle \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \rangle = x_1 y_1 + x_2 y_2$ is the standard scalar product and $\mathrm{sgn}(t) = \begin{cases} 1, & t>0, \\ 0, & t=0, \\ -1, & t<0. \end{cases}$