1. Is the function $G(x,y) = \mathbf{1}_{\{x+y\geq 0\}}$ the distribution function of some random vector in $\mathbb{R}^2$? Explain.

2. Find a constant C such that $f\colon \mathbb{R}^2 \to \mathbb{R}$ given as $f(x,y) = \frac{C}{(1+x^2+y^2)^{3/2}}$ is a density function. Show that both marginals have the Cauchy distribution.

3. Let $X_1, X_2, \ldots$ be independent exponential random variables with parameter 1. Show that for every n, the distribution of $X_1 + \ldots + X_n$ is Gamma(n).

4. Let (X,Y) be a random vector in $\mathbb{R}^2$ with density $f(x,y) = cxy\mathbf{1}_{0<x<y<1}$. Find c and $\mathbb{P}\left(X+Y<1\right)$. Are X and Y independent? Find the density of $(X/Y, Y)$. Are X/Y and Y independent? What is the conditional density of X given $Y = y$?

5. Let X and Y be independent standard Gaussian random variables. Show that X/Y has the Cauchy distribution. Find $\mathbb{P}\left(X^2+Y^2<a\right)$ for $a>0$ and $\mathbb{E}\sqrt{X^2+Y^2}$.

6. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ uniformly distributed on the simplex $\{x \in \mathbb{R}^n,\ x_1 + \ldots + x_n \leq 1, x_1, \ldots, x_n \geq 0\}$. Find $\mathbb{E}X_1$, $\mathbb{E}X_1^2$, $\mathbb{E}X_1X_2$, the covariance matrix of X and its determinant for a) $n=2$ and $n=3$ b)* any $n\geq 2$.

7. Let X be a nonnegative continuous random variable such that $\mathbb{E}X < \infty$. Show that

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\left(X>t\right) \mathrm{d}t.$$

8. Let $U_1, \ldots, U_n$ be a sequence of i.i.d. random variables, each uniform on $[0,1]$. Let $U_1^*, \ldots, U_n^*$ be its nondecreasing rearrangement, that is $U_1^* \leq \ldots \leq U_n^*$. In particular, $U_1^* = \min\{U_1, \ldots, U_n\}$ and $U_n^* = \max\{U_1, \ldots, U_n\}$. Find a) $\mathbb{E}U_1^*$ and $\mathbb{E}U_n^*$, b)* $\mathbb{E}U_k^*$.

9.* Show the *lack of memory* property characterises the exponential distribution. Specifically, let X be a random variable such that for every positive s and t, $\mathbb{P}\left(X>s\right)>0$ and $\mathbb{P}\left(X>s+t|X>s\right) = \mathbb{P}\left(X>t\right)$. Show that X has the exponential distribution.

10.* Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs. Show that the series $\sum_{n=1}^\infty \varepsilon_n/2^n$ defines a random variable which is uniform on $[-1,1]$.