Probability 21-325 **Homework 6 (due 27st Feb)** **TT**

1. Let X and Y be independent exponential random variables with parameters λ and μ. Show that $\min\{X, Y\}$ has the exponential distribution with parameter $\lambda + \mu$.

2. If X has the exponential distribution, show the *lack of memory* property: for every positive s and t,

$$\mathbb{P}(X > s + t | X > s) = \mathbb{P}(X > t).$$

3. Let $X_1, \ldots, X_n$ be independent exponential random variables with parameter 1. Find the distribution function of $Y_n = \max\{X_1, \ldots, X_n\}$. What is the pointwise limit of the distribution function F_n of $Y_n - \log n$? Is the limiting function a distribution function?

4. Let $X_1, X_2, \ldots$ be i.i.d. continuous random variables. Define N as the unique index such that

$$X_1 \geq X_2 \geq \ldots \geq X_{N-1} \text{ and } X_{N-1} < X_N.$$

Prove that $\mathbb{P}(N = k) = (k-1)/k!$, $k = 1, 2, \ldots$ and find $\mathbb{E}N$.

5.* Let X be a standard Gaussian random variable and Y be an exponential random variable with parameter 1. Show that $\sqrt{2Y}X$ has the symmetric (two-sided) exponential distribution with parameter 1.

— Revision problems before Midterm 1 (not for grading) —

1. We randomly put 20 identical balls into 7 labelled boxes (multiple occupancies are allowed). What is the probability that at least one box is empty?

2. A fair die is thrown 4 times. For $i = 1, \ldots, 6$, let X_i be a random variable equal to 1 if the side showing "i" was rolled at least once and equal to 0 otherwise. What is the distribution of X_i called? Find the expectation of X_i. Let X be the number of different outcomes obtained (for example, if the outcomes are $4, 6, 6, 1$, then $X = 3$). Express X in terms of the X_i. Find the expectation and variance of X.

3. Let $A_1, \ldots, A_n$ be events such that $\mathbb{P}(A_k) = 1 - \frac{1}{2^k}$ for each $k = 1, \ldots, n$. Show that the set $\bigcap_{k=1}^n A_k$ is nonempty.

4. We toss repeatedly a fair coin. Let X be the number of tosses until 3 heads have occurred in a row. Find the expectation of X.

5. What is the distribution of the sum of two independent $\mathrm{Bin}(m, p)$ and $\mathrm{Bin}(n, p)$ random variables?

6. We pick independently uniformly at random two numbers from $\{1, 2, \ldots, 2n\}$. What is the expectation of the smaller one?

7. A standard deck of 52 cards is shuffled and we remove cards from the top of the pile one by one and put aside without checking what the removed cards are. We look at 13th card. What is the probability that it is a spade?

8. What is the probability that the quadratic equation $x^2 + 2bx + c = 0$ has real roots?

9. Three dice are thrown. What is more likely: getting the sum 11 or 12?

10. What is the expected number of throws of a fair die until one gets a 6?

11. You are dealt a bridge hand (13 cards from a regular deck of 52 cards). What is the probability of getting all cards of the same suit?

12. There are 20 interesting restaurants in Pittsburgh. For 7 consecutive days, every night we dine at a randomly selected interesting restaurant. What is the expected number of restaurants we have tried?

13. In a country where each family wants a girl, each family continues having babies till they have a baby girl. After some time, what is the proportion of boys to girls in the country? (Assume that the probability of having a boy or a girl is the same.)

14. A person dies, and he arrives at the gate to heaven. There are 3 doors in the heaven. One of the door leads to heaven, second one leads to a 1 day stay at hell and then back to the gate and the third one leads to a 2 day stay at hell and then back to the gate. Every time the person is back at the gate, the 3 doors are reshuffled. How long will it take the person to reach heaven?